# STUDIES ON THE GARNIER SYSTEM IN TWO VARIABLES

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Abstract. We study some Hamiltonian structures of the Garnier system in two variables from the viewpoints of its symmetry and holomorphy properties. We also give a generalization of *Okamoto transformation* of the sixth Painlevé system.

## 1. Introduction

In this paper, we present a new symmetry of the Garnier system in two variables $t, s$ with the Hamiltonians $H_1, H_2$ (see (11) in Section 3) explicitly given by

$$S:(q_1,p_1,q_2,p_2,t,s) \rightarrow \left(q_1+\frac{q_2p_2+\alpha_1}{p_1},p_1,\frac{p_2}{p_1},-q_2p_1,t,\frac{t}{s}\right). \tag{1}$$

The transformation $S$ is birational and symplectic, but this system is not invariant under $S$.

By using (1), we obtain a new expression of this system explicitly given as follows:

$$\begin{aligned}
&dq_1=\frac{\partial H_1'}{\partial p_1}dt+\frac{\partial H_2'}{\partial p_1}ds, \quad dp_1=-\frac{\partial H_1'}{\partial q_1}dt-\frac{\partial H_2'}{\partial q_1}ds,\\
&dq_2=\frac{\partial H_1'}{\partial p_2}dt+\frac{\partial H_2'}{\partial p_2}ds, \quad dp_2=-\frac{\partial H_1'}{\partial q_2}dt-\frac{\partial H_2'}{\partial q_2}ds,\\
&H_1'=H_{VI}(q_1,p_1,t;\alpha_1+\alpha_4+\alpha_6,-\alpha_1-\alpha_2,\alpha_2,\alpha_1+\alpha_5,\alpha_1+\alpha_3)\\
&\quad+\alpha_2\{\frac{(s-1)q_2}{(t-1)(t-s)}-\frac{sq_1}{t(t-s)}+\frac{q_1q_2}{t(t-1)}\}p_2+\alpha_4\frac{(q_1-q_2)p_1}{t-s}\\
&\quad+\{\frac{2(s-1)q_1q_2}{(t-1)(t-s)}-\frac{tq_2^2+sq_1^2}{t(t-s)}+\frac{(q_1^2+t)q_2}{t(t-1)}\}p_1p_2,\\
&H_2'=\pi(H_1') \quad (2\alpha_1+\alpha_2+\cdots+\alpha_6=1),
\end{aligned} \tag{2}$$

where the transformation $\pi$ is given by

(3)

$$\pi:(q_1,p_1,q_2,p_2,t,s;\alpha_1,\alpha_2,\dots,\alpha_6) \rightarrow (q_2,p_2,q_1,p_1,s,t;\alpha_1,\alpha_4,\alpha_3,\alpha_2,\alpha_5,\alpha_6).$$

The symbol $H_{VI}$ denotes the Hamiltonian of $P_{VI}$ (see in Section 2). Comparing with the original ones, the form of each transformed Hamiltonian $H_i'$ $(i=1,2)$ is $H_{VI}$ with new parameters in addition to new interaction term $R_i'$ $(i=1,2)$.

**Remark 1.1.** On the polynomiality of the transformed Hamiltonian by (1), the change of time-variables $t, s$ of this transformation is not essential. In fact, for the

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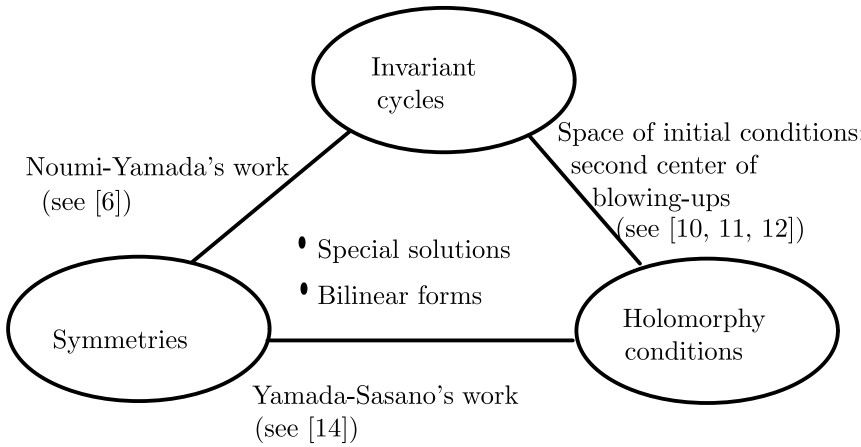


FIGURE 1

transformation without changing time-variables $t, s$

$$S' : (q_1, p_1, q_2, p_2) \rightarrow \left(q_1 + \frac{q_2 p_2 + \alpha_1}{p_1}, p_1, \frac{p_2}{p_1}, -q_2 p_1\right), \tag{4}$$

we can obtain a polynomial Hamiltonian.

As these results, for (2) we will obtain

(1) Different type of holomorphy conditions from the Garnier system (see Section 5).
(2) Different type of invariant cycles from the Garnier system (see Section 8).
(3) New Dynkin diagram (see Figure 1).
(4) New expression of birational symmetries with a natural extension of *Okamoto transformation* (see in Section 10).
(5) Some algebraic solutions including an extension of *Umemura's solution* of $P_{VI}$ (see in Section 11).

We also study the Garnier system in two variables from the viewpoint of its holomorphy and symmetry.

This paper is organized as follows. In Section 1, we review the sixth Painlevé system from the viewpoint of its holomorphy and symmetry. In section 2, we recall the Hamiltonians of the Garnier system in two variables and find the holomorphy conditions of this system. In Section 3, we will consider the relation between the holomorphy conditions with the accessible singularities of this system. In Section 4, we propose some new expressions of this system and generalizations of *Okamoto transformation* of the sixth Painlevé system. In Section 5, we will consider the relation between the holomorphy conditions with the accessible singularities of each system given in Section 4. In Section 6, we study some Bäcklund transformations of the Garnier system in two variables. In Section 7, we study some Bäcklund transformations of (2). In the final section, we study some algebraic solutions for (2) including an extension of *Umemura's solution* of $P_{VI}$.

## 2. Review of $P_{VI}$-case

As is well-known, each of the Painlevé equations $P_J$ $(J = VI, V, IV, III, II, I)$ is equivalent to a polynomial Hamiltonian system $H_J$. In 1997, K. Takano et

al. studied some Hamiltonian structures of Painlevé systems except for the first Painlevé system (see [15]). They showed that each space of initial conditions of $H_J$ is obtained by gluing a finite number of copies of $\mathbb{C}^2$ via the birational and symplectic transformations. In each coordinate system, the Hamiltonian is expressed as a *polynomial* of the canonical coordinates. Moreover, they showed that by using these patching data in the phase space of each $H_J$, each Hamiltonian $H_J$ is uniquely determined. For example, let us review the case of the sixth Painlevé system. The sixth Painlevé system can be written as the Hamiltonian system

(5)
$$\begin{aligned}
&\frac{dq}{dt}=\frac{\partial H_{VI}}{\partial p}, \quad \frac{dp}{dt}=-\frac{\partial H_{VI}}{\partial q},\\
&H_{VI}(q,p,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)\\
&=\frac{1}{t(t-1)}[p^2(q-t)(q-1)q-\{(\alpha_0-1)(q-1)q+\alpha_3(q-t)q\\
&+\alpha_4(q-t)(q-1)\}p+\alpha_2(\alpha_1+\alpha_2)(q-t)] \quad (\alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_4=1).
\end{aligned}$$

By the work of Okamoto, it is known that the system (5) has the affine Weyl group symmetry of type $D_4^{(1)}$, whose generators $s_i,\ i=0,1,2,3,4$, are given by

(6)
$$\begin{aligned}
s_0(q,p,t;\alpha_0,\alpha_1,\dots,\alpha_4) \to&(q,p-\frac{\alpha_0}{q-t},p,t;-\alpha_0,\alpha_1,\alpha_2+\alpha_0,\alpha_3,\alpha_4),\\
s_1(q,p,t;\alpha_0,\alpha_1,\dots,\alpha_4) \to&(q,p,t;\alpha_0,-\alpha_1,\alpha_2+\alpha_1,\alpha_3,\alpha_4),\\
s_2(q,p,t;\alpha_0,\alpha_1,\dots,\alpha_4) \to&(q+\frac{\alpha_2}{p},p,t;\alpha_0+\alpha_2,\alpha_1+\alpha_2,-\alpha_2,\alpha_3+\alpha_2,\alpha_4+\alpha_2),\\
s_3(q,p,t;\alpha_0,\alpha_1,\dots,\alpha_4) \to&(q,p-\frac{\alpha_3}{q-1},t;\alpha_0,\alpha_1,\alpha_2+\alpha_3,-\alpha_3,\alpha_4),\\
s_4(q,p,t;\alpha_0,\alpha_1,\dots,\alpha_4) \to&(q,p-\frac{\alpha_4}{q},t;\alpha_0,\alpha_1,\alpha_2+\alpha_4,\alpha_3,-\alpha_4).
\end{aligned}$$

The list (6) should be read as

$$\begin{aligned}
&s_0(q)=q, \quad s_0(p)=p-\frac{\alpha_0}{q-t}, \quad s_0(t)=t,\\
&s_0(\alpha_0)=-\alpha_0, \quad s_0(\alpha_1)=\alpha_1, \quad s_0(\alpha_2)=\alpha_2+\alpha_0,\\
&s_0(\alpha_3)=\alpha_3, \quad s_0(\alpha_4)=\alpha_4.
\end{aligned}$$

The Hamiltonian $H_{VI}$ (5) is a polynomial in the variables $q,p$. In this sense we call the system (5) as a polynomial Hamiltonian system. Consider the following

birational and symplectic transformations $r_i,\ i=0,1,2,3,4$:

$$
\begin{aligned}
&r_0 : x_0 = -((q-t)p-\alpha_0)p,\ y_0 = \frac{1}{p},\\
&r_1 : x_1 = \frac{1}{q},\ y_1 = -(qp+\alpha_1+\alpha_2)q,\\
&r_2 : x_2 = \frac{1}{q},\ y_2 = -(qp+\alpha_2)q,\\
&r_3 : x_3 = -((q-1)p-\alpha_3)p,\ y_3 = \frac{1}{p},\\
&r_4 : x_4 = -(qp-\alpha_4)p,\ y_4 = \frac{1}{p}.
\end{aligned} \tag{7}
$$

Since the transformations $r_i$ are symplectic, the system (5) is transformed into a Hamiltonian system, whose Hamiltonian may have poles. It is remarkable that the transformed system becomes again a polynomial Hamiltonian system for any $i=0,1,\dots,4$. Furthermore, this holomorphy property uniquely characterizes the system (5):

**Proposition 2.1.** *Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in \mathbb{C}(t)[q,p]$. We assume that*

*(A1) $deg(H)=5$ with respect to $q,p$.*

*(A2) This system becomes again a polynomial Hamiltonian system in each coordinate system $r_i$ $(j=0,1,\dots,4)$.*

*Then such a system coincides with the system* (5).

In this paper, we call the conditions $r_i$ $(j=0,1,..,4)$ *holomorphy conditions* of the sixth Painlevé system. The space of initial conditions of $P_{VI}$ is covered by these coordinate systems. Each coordinate system contains a one-parameter family of meromorphic solutions of this system.

**Remark 2.2.** If we look for a polynomial Hamiltonian system which admits the symmetry (6), we have to consider huge polynomial in variables $q,p,t,\alpha_i$. On the other hand, in the holomorphy requirement (7), we only need to consider polynomials in $q,p$. This reduces the number of unknown coefficients drastically.

## 3. Holomorphy condisions of the Garnier system in two variables

Consider a Fuchsian differential equation on $\mathbb{P}^1$

$$
\frac{d^2Y}{dZ^2}+P_1(Z)\frac{dY}{dZ}+P_2(Z)Y=0 \tag{8}
$$

with regular singularities $Z=0, Z=1, Z=t, Z=s, Z=\infty$, apparent singularities $Z=q_1, Z=q_2$ and the Riemann scheme

$$
\begin{pmatrix}
Z=0 & Z=1 & Z=t & Z=s & Z=q_1 & Z=q_2 & Z=\infty\\
0 & 0 & 0 & 0 & 0 & 0 & \alpha_1\\
\alpha_6 & \alpha_5 & \alpha_3 & \alpha_4 & 2 & 2 & \alpha_1+\alpha_2
\end{pmatrix} \tag{9}
$$

assuming that the Fuchs relation

$$
2\alpha_1+\alpha_2+\cdots+\alpha_6=1 \tag{10}
$$

is satisfied. The monodromy preserving deformations of the equation (8) with the scheme (9) is described as a completely integrable Hamiltonian system. By using

a transformation (see [18]), we obtain the Garnier system in two variables, which is equivalent to the Hamiltonian system given by (see [18], cf. [3])

(11)
$$\begin{aligned}
&dq_1=\frac{\partial H_1}{\partial p_1}dt+\frac{\partial H_2}{\partial p_1}ds, \quad dp_1=-\frac{\partial H_1}{\partial q_1}dt-\frac{\partial H_2}{\partial q_1}ds,\\
&dq_2=\frac{\partial H_1}{\partial p_2}dt+\frac{\partial H_2}{\partial p_2}ds, \quad dp_2=-\frac{\partial H_1}{\partial q_2}dt-\frac{\partial H_2}{\partial q_2}ds,\\
&H_1=H_{VI}(q_1,p_1,t;\alpha_4+\alpha_6,\alpha_2,\alpha_1,\alpha_5,\alpha_3)\\
&\quad+(2\alpha_1+\alpha_2)\frac{q_1q_2p_2}{t(t-1)}+\alpha_3\{\frac{p_1}{t-s}-\frac{(s-1)p_2}{(t-s)(t-1)}\}q_2+\alpha_4\frac{s(p_2-p_1)q_1}{t(t-s)}\\
&\quad+\{\frac{2(s-1)p_1p_2}{(t-s)(t-1)}-\frac{tp_1^2+sp_2^2}{t(t-s)}+\frac{(2q_1p_1+q_2p_2)p_2}{t(t-1)}\}q_1q_2,\\
&H_2=\pi(H_1),
\end{aligned}$$

where the transformation $\pi$ is explicitly given by

$$\begin{aligned}
\pi:&(q_1,p_1,q_2,p_2,t,s;\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6)\\
&\to(q_2,p_2,q_1,p_1,s,t;\alpha_1,\alpha_2,\alpha_4,\alpha_3,\alpha_5,\alpha_6).
\end{aligned} \tag{12}$$

The symbol $H_{VI}$ is given by (5) in Section 0. We remark that Kimura and Okamoto introduced polynomial Hamiltonians for the Garnier system (see [3]). After that, by addition to improve, finally it has been of the form $H_1, H_2$ by T. Tsuda (see [22, 23, 24, 25]).

Here we recall the definition of a symplectic transformation and its properties (see [15]). Let

$$\varphi: x=x(X,Y,Z,W,t),\ y=y(X,Y,Z,W,t),\ z=z(X,Y,Z,W,t),$$

$$w=w(X,Y,Z,W,t),\ t=t$$

be a biholomorphic mapping from a domain $D$ in $\mathbb{C}^5 \ni (X,Y,Z,W,t)$ into $\mathbb{C}^5 \ni (x,y,z,w,t)$. We say that the mapping is symplectic if

$$dx\wedge dy+dz\wedge dw=dX\wedge dY+dZ\wedge dW,$$

where $t$ is considered as a constant or a parameter, namely, if, for $t=t_0$, $\varphi_{t_0}=\varphi|_{t=t_0}$ is a symplectic mapping from the $t_0$-section $D_{t_0}$ of $D$ to $\varphi(D_{t_0})$. Suppose that the mapping is symplectic. Then any Hamiltonian system

$$dx/dt=\partial H/\partial y,\quad dy/dt=-\partial H/\partial x,\quad dz/dt=\partial H/\partial w,\quad dw/dt=-\partial H/\partial z$$

is transformed to

$$dX/dt=\partial K/\partial Y,\quad dY/dt=-\partial K/\partial X,\quad dZ/dt=\partial K/\partial W,\quad dW/dt=-\partial K/\partial Z,$$

where

$$(A)\quad dx\wedge dy+dz\wedge dw-dH\wedge dt=dX\wedge dY+dZ\wedge dW-dK\wedge dt.$$

Here $t$ is considered as a variable. By this equation, the function $K$ is determined by $H$ uniquely modulo functions of $t$, namely, modulo functions independent of $X, Y, Z$ and $W$.

**Theorem 3.1.** *Let us consider a polynomial Hamiltonian system with Hamiltonians $H_i \in \mathbb{C}(t,s)[q_1,p_1,q_2,p_2]$ $(i=1,2)$. We assume that*

*(A1) $deg(H_i)=5$ with respect to $q_1,p_1,q_2,p_2$.*

*(A2) This system becomes again a polynomial Hamiltonian system in each coordinate $r_i,\ i=1,2,\dots,6$:*

(13)
$$\begin{aligned}
&r_1: x_1=\frac{1}{q_1},\ y_1=-q_1(q_1p_1+q_2p_2+\alpha_1),\ z_1=\frac{q_2}{q_1},\ w_1=q_1p_2,\\
&r_2: x_2=\frac{1}{q_1},\ y_2=-q_1(q_1p_1+q_2p_2+\alpha_1+\alpha_2),\ z_2=\frac{q_2}{q_1},\ w_2=q_1p_2,\\
&r_3: x_3=-p_1(q_1p_1-\alpha_3),\ y_3=\frac{1}{p_1},\ z_3=q_2,\ w_3=p_2,\\
&r_4: x_4=q_1,\ y_4=p_1,\ z_4=-p_2(q_2p_2-\alpha_4),\ w_4=\frac{1}{p_2},\\
&r_5: x_5=-((q_1+q_2-1)p_1-\alpha_5)p_1,\ y_5=\frac{1}{p_1},\ z_5=q_2,\ w_5=p_2-p_1,\\
&r_6: x_6=-((q_1+tq_2/s-t)p_1-\alpha_6)p_1,\ y_6=\frac{1}{p_1},\ z_6=q_2,\ w_6=p_2-\frac{tp_1}{s}.
\end{aligned}$$

*Then such a system coincides with the system* (11).

We remark that each transformation of each coordinate $r_i,\ i=1,2,\dots,6$, is birational and symplectic. These transformations are appeared as the patching data in the space of initial conditions of the system (11). On the construction of these transformations $r_i$ $(i=1,2,\dots,6)$, we will explain in the next section.

**Proposition 3.2.** *In each coordinate $r_i,\ i=1,2,\dots,6$, the Hamiltonians $H_{j1}$ and $H_{j2}$ on $U_j\times B$ are expressed as a polynomial in $x_j,y_j,z_j,w_j$ and a rational function in $t$ and $s$, and satisfy the following conditions*:

(14)
$$\begin{aligned}
&dq_1\wedge dp_1+dz\wedge dp_2-dH_1\wedge dt-dH_2\wedge ds\\
&=dx_j\wedge dy_j+dz_j\wedge dw_j-dH_{j1}\wedge dt-dH_{j2}\wedge ds \quad (j=1,2,\dots,5),\\
&dq_1\wedge dp_1+dq_2\wedge dp_2-d(H_1-(1-q_2/s)p_1)\wedge dt-d(H_2-(1-q_1/t)p_2)\wedge ds\\
&=dx_6\wedge dy_6+dz_6\wedge dw_6-dH_{61}\wedge dt-dH_{62}\wedge ds.
\end{aligned}$$

**Proof of Theorem 3.1.** The polynomial $H$ satisfying $(A1)$ has 126 unknown coefficients in $\mathbb{C}(t,s)$. At first, resolving the coordinate $r_1$ in the variables $q_1,p_1,q_2,p_2$, we obtain

$(R_1)$
$$q_1=1/x_1,\quad p_1=-(x_1y_1+z_1w_1+\alpha_1)x_1,\quad q_2=z_1/x_1,\quad p_2=w_1x_1.$$

By $R_1$, we transform $H$ into $R_1(H)$, which has poles in only $x_1$. For $R_1(H)$, we only have to determine the unknown coefficients so that they cancel the poles of $R_1(H)$.

For the transformations $r_i,(i=2,3,4,5)$, we can repeat the same way.

We must be careful of the case of the transformation $r_6$. The relation between the coordinate system $(x_6,y_6,z_6,w_6)$ and the coordinate system $(q_1,p_1,q_2,p_2)$ is

given by

(15)
$$\begin{aligned}&dx_6 \wedge dy_6 + dz_6 \wedge dw_6\\&= dq_1 \wedge dp_1 + dq_2 \wedge dp_2 + d((1-q_2/s)p_1) \wedge dt + d((1-q_1/t)p_2) \wedge ds.\end{aligned}$$

Resolving the coordinate $r_6$ in the variables $q_1, p_1, q_2, p_2$, we obtain

$(R_6)$
$$q_1 = t - x_6 y_6^2 + \alpha_6 y_6 - \frac{tz_6}{s}, \quad p_1 = 1/y_6, \quad q_2 = z_6, \quad p_2 = w_6 + \frac{t}{sy_6}.$$

In this case, we must consider the polynomiality for $R_6(H_1 - (1-q_2/s)p_1)$ and $R_6(H_2 - (1-q_1/t)p_2)$, respectively.

In this way, we can obtain the Hamiltonians $H_1, H_2$. ∎

## 4. On some Hamiltonian structures of the system (11)

In this section, we will consider the relation between the holomorphy conditions $r_j$ with the accessible singularities of the system (11). Let us take the compactification

$$(q_1, p_1, q_2, p_2, t, s) \in \mathbb{C}^4 \times B_2 \text{ to } ([z_0 : z_1 : z_2 : z_3 : z_4], t, s) \in \mathbb{P}^4 \times B_2$$

with the natural embedding

$$(q_1, p_1, q_2, p_2) = (z_1/z_0, z_2/z_0, z_3/z_0, z_4/z_0).$$

Here $B_2 = \mathbb{C}^2 - \{t(t-1)s(s-1) = 0\}$. Fixing the parameters $\alpha_i$, consider the product $\mathbb{P}^4 \times B_2$ and extend the regular vector field on $\mathbb{C}^4 \times B_2$ to a rational vector field $\tilde{v}$ on $\mathbb{P}^4 \times B_2$. It is easy to see that $\mathbb{P}^4$ is covered by five copies of $\mathbb{C}^4$:

$$U_0 = \mathbb{C}^4 \ni (q_1, p_1, q_2, p_2), \ U_j = \mathbb{C}^4 \ni (X_j, Y_j, Z_j, W_j) \ (j = 1, 2, 3, 4),$$

via the following rational transformations

$$\begin{aligned}&1) X_1 = 1/q_1, \quad Y_1 = p_1/q_1, \quad Z_1 = q_2/q_1, \quad W_1 = p_2/q_1,\\&2) X_2 = q_1/q_2, \quad Y_2 = p_1/q_2, \quad Z_2 = 1/q_2, \quad W_2 = p_2/q_2,\\&3) X_3 = q_1/p_1, \quad Y_3 = 1/p_1, \quad Z_3 = q_2/p_1, \quad W_3 = p_2/p_1,\\&4) X_4 = q_1/p_2, \quad Y_4 = p_1/p_2, \quad Z_4 = q_2/p_2, \quad W_4 = 1/p_2.\end{aligned} \tag{16}$$

By the following lemma, we will show that each coordinate system $(x_i, y_i, z_i, w_i)$ $(i = 1, 2, \ldots, 6)$ can be obtained by successive blowing-up procedures of the accessible singularities in the boundary divisor $\mathcal{H}(\cong \mathbb{P}^3) \subset \mathbb{P}^4$.

**Proposition 4.1.** *By resolving the following accessible singularities in the boundary divisor $\mathcal{H}(\cong \mathbb{P}^3) \subset \mathbb{P}^4$*

$$\begin{cases} C_1 = \{(X_1, Y_1, Z_1, W_1) | X_1 = Y_1 = W_1 = 0\},\\ C_2 = \{(X_2, Y_2, Z_2, W_2) | Y_2 = Z_2 = W_2 = 0\},\\ P_3 = \{(X_3, Y_3, Z_3, W_3) | X_3 = Y_3 = Z_3 = W_3 = 0\},\\ P_4 = \{(X_4, Y_4, Z_4, W_4) | X_4 = Y_4 = Z_4 = W_4 = 0\},\\ P_5 = \{(X_3, Y_3, Z_3, W_3) | X_3 = Y_3 = Z_3 = 0, W_3 = 1\},\\ P_6 = \{(X_3, Y_3, Z_3, W_3) | X_3 = Y_3 = Z_3 = 0, W_3 = t/s\},\end{cases}$$

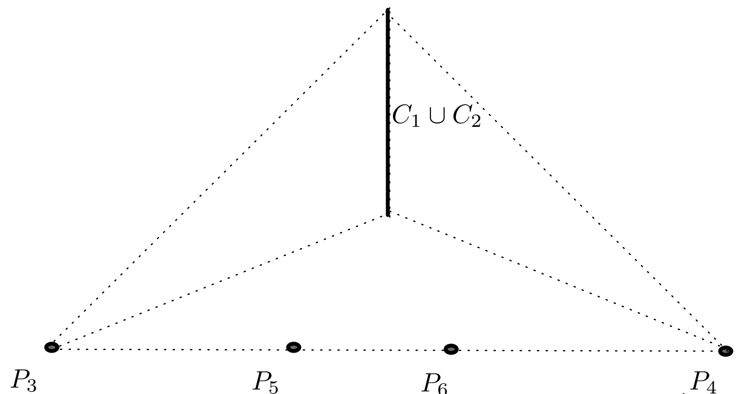


FIGURE 2. The figure denotes the boundary divisor $\mathcal{H}$ in $\mathbb{P}^4$. The dark parts correspond to the accessible singularities of the system (11).

*we can obtain the coordinate systems* $(x_i, y_i, z_i, w_i)$ $(i = 1, 2, \dots, 6)$. *Here* $C_1 \cup C_2 \cong \mathbb{P}^1$.

**Proof of Proposition 4.1.** At first, we give an explicit resolution process for the accessible singularity $C_1 \cup C_2 \cong \mathbb{P}^1$ by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singularity $C_1 \cup C_2 \cong \mathbb{P}^1$.

**Step 1:** We blow up along the curve $C_1 \cup C_2 \cong \mathbb{P}^1$:

$$x_1^{(1)} = X_1 \ , \quad y_1^{(1)} = \frac{Y_1}{X_1} \ , \quad z_1^{(1)} = Z_1 \ , \quad w_1^{(1)} = \frac{W_1}{X_1},$$
$$x_2^{(1)} = X_2 \ , \quad y_2^{(1)} = \frac{Y_2}{Z_2} \ , \quad z_2^{(1)} = Z_2 \ , \quad w_2^{(1)} = \frac{W_2}{Z_2}.$$

**Step 2:** We blow up along the curve

$$\{(x_1^{(1)}, y_1^{(1)}, z_1^{(1)}, w_1^{(1)}) | x_1^{(1)} = y_1^{(1)} = w_1^{(1)} = 0\} \cup$$
$$\{(x_2^{(1)}, y_2^{(1)}, z_2^{(1)}, w_2^{(1)}) | y_2^{(1)} = z_2^{(1)} = w_2^{(1)} = 0\} \cong \mathbb{P}^1:$$

$$x_1^{(2)} = x_1^{(1)} \ , \quad y_1^{(2)} = \frac{y_1^{(1)}}{x_1^{(1)}} \ , \quad z_1^{(2)} = z_1^{(1)} \ , \quad w_1^{(2)} = \frac{w_1^{(1)}}{x_1^{(1)}},$$
$$x_2^{(2)} = x_2^{(1)} \ , \quad y_2^{(2)} = \frac{y_2^{(1)}}{z_2^{(1)}} \ , \quad z_2^{(2)} = z_2^{(1)} \ , \quad w_2^{(2)} = \frac{w_2^{(1)}}{z_2^{(1)}}.$$

It is easy to see that there are two accessible singularities

$$\begin{aligned}
S_1 =&\{(x_1^{(2)}, y_1^{(2)}, z_1^{(2)}, w_1^{(2)}) | x_1^{(2)} = y_1^{(2)} + z_1^{(2)} w_1^{(2)} + \alpha_1 = 0\} \cup \\
&\{(x_2^{(2)}, y_2^{(2)}, z_2^{(2)}, w_2^{(2)}) | z_2^{(2)} = w_2^{(2)} + x_2^{(2)} y_2^{(2)} + \alpha_1 = 0\}, \\
S_2 =&\{(x_1^{(2)}, y_1^{(2)}, z_1^{(2)}, w_1^{(2)}) | x_1^{(2)} = y_1^{(2)} + z_1^{(2)} w_1^{(2)} + \alpha_1 + \alpha_2 = 0\} \cup \\
&\{(x_2^{(2)}, y_2^{(2)}, z_2^{(2)}, w_2^{(2)}) | z_2^{(2)} = w_2^{(2)} + x_2^{(2)} y_2^{(2)} + \alpha_1 + \alpha_2 = 0\}.
\end{aligned}$$

**Step 3:** We blow up along the surface

$$\begin{aligned}S_1 =&\{(x_1^{(2)},y_1^{(2)},z_1^{(2)},w_1^{(2)})|x_1^{(2)}=y_1^{(2)}+z_1^{(2)}w_1^{(2)}+\alpha_1=0\} \cup \\ &\{(x_2^{(2)},y_2^{(2)},z_2^{(2)},w_2^{(2)})|z_2^{(2)}=w_2^{(2)}+x_2^{(2)}y_2^{(2)}+\alpha_1=0\}:\end{aligned}$$

$$x_1^{(3)}=x_1^{(2)} \ , \quad y_1^{(3)}=\frac{y_1^{(2)}+z_1^{(2)}w_1^{(2)}+\alpha_1}{x_1^{(2)}} \ , \quad z_1^{(3)}=z_1^{(2)} \ , \quad w_1^{(3)}=w_1^{(2)},$$

$$x_2^{(3)}=x_2^{(2)} \ , \quad y_2^{(3)}=y_2^{(2)} \ , \quad z_2^{(3)}=z_2^{(2)} \ , \quad w_2^{(3)}=\frac{w_2^{(2)}+x_2^{(2)}y_2^{(2)}+\alpha_1}{z_2^{(2)}}.$$

**Step 4:** We blow up along the surface

$$\begin{aligned}S_2 =&\{(x_1^{(2)},y_1^{(2)},z_1^{(2)},w_1^{(2)})|x_1^{(2)}=y_1^{(2)}+z_1^{(2)}w_1^{(2)}+\alpha_1+\alpha_2=0\} \cup \\ &\{(x_2^{(2)},y_2^{(2)},z_2^{(2)},w_2^{(2)})|z_2^{(2)}=w_2^{(2)}+x_2^{(2)}y_2^{(2)}+\alpha_1+\alpha_2=0\}:\end{aligned}$$

$$x_1^{(4)}=x_1^{(2)} \ , \quad y_1^{(4)}=\frac{y_1^{(2)}+z_1^{(2)}w_1^{(2)}+\alpha_1+\alpha_2}{x_1^{(2)}} \ , \quad z_1^{(4)}=z_1^{(2)} \ , \quad w_1^{(4)}=w_1^{(2)},$$

$$x_2^{(4)}=x_2^{(2)} \ , \quad y_2^{(4)}=y_2^{(2)} \ , \quad z_2^{(4)}=z_2^{(2)} \ , \quad w_2^{(4)}=\frac{w_2^{(2)}+x_2^{(2)}y_2^{(2)}+\alpha_1+\alpha_2}{z_2^{(2)}}.$$

We have resolved the accessible singularity $C_1 \cup C_2 \cong \mathbb{P}^1$.

By choosing new coordinate systems as

$$(x_i,y_i,z_i,w_i)=(x_1^{(i+2)},-y_1^{(i+2)},z_1^{(i+2)},w_1^{(i+2)}) \ (i=1,2),$$

we can obtain the coordinate systems $(x_i,y_i,z_i,w_i)$ $(i=1,2)$ given in Theorem 3.1.

Next, we give an explicit resolution process for the accessible singular point $P_3$ by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singular point $P_3$.

**Step 1:** We blow up at the point $P_3$:

$$x_3^{(1)}=\frac{X_3}{Y_3} \ , \quad y_3^{(1)}=Y_3 \ , \quad z_3^{(1)}=\frac{Z_3}{Y_3} \ , \quad w_3^{(1)}=\frac{W_3}{Y_3}.$$

**Step 2:** We blow up along the surface

$$\{(x_3^{(1)},y_3^{(1)},z_3^{(1)},w_3^{(1)})|x_3^{(1)}=y_3^{(1)}=0\}:$$

$$x_3^{(2)}=\frac{x_3^{(1)}}{y_3^{(1)}} \ , \quad y_3^{(2)}=y_3^{(1)} \ , \quad z_3^{(2)}=z_3^{(1)} \ , \quad w_3^{(2)}=w_3^{(1)}.$$

**Step 3:** We blow up along the surface

$$\{(x_3^{(2)},y_3^{(2)},z_3^{(2)},w_3^{(2)})|x_3^{(2)}-\alpha_3=y_3^{(2)}=0\}:$$

$$x_3^{(3)}=\frac{x_3^{(2)}-\alpha_3}{y_3^{(2)}} \ , \quad y_3^{(3)}=y_3^{(2)} \ , \quad z_3^{(3)}=z_3^{(2)} \ , \quad w_3^{(3)}=w_3^{(2)}.$$

We have resolved the accessible singular point $P_3$.

By choosing a new coordinate system as

$$(x_3,y_3,z_3,w_3)=(-x_3^{(3)},y_3^{(3)},z_3^{(3)},w_3^{(3)}),$$

we can obtain the coordinate system $(x_3,y_3,z_3,w_3)$ given in Theorem 3.1.

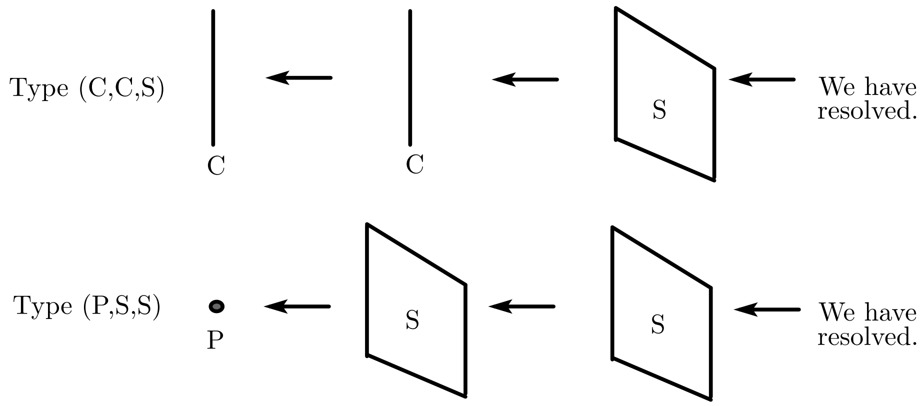


FIGURE 3. This figure denotes resolution process of the accessible singularities. We denote that P is a point, C is a curve and S is a surface.

Next, we give an explicit resolution process for the accessible singular point $P_6$ by giving a convenient coordinate system at each step.

By the following steps, we can resolve the accessible singular point $P_6$.

**Step 0:** We take the coordinate system centered at $P_6$:

$$x_6^{(0)} = X_3 \ , \quad y_6^{(0)} = Y_3 \ , \quad z_6^{(0)} = Z_3 \ , \quad w_6^{(0)} = W_3 - t/s.$$

**Step 1:** We blow up at the point $P_6$:

$$x_6^{(1)} = \frac{x_6^{(0)}}{y_6^{(0)}} \ , \quad y_6^{(1)} = y_6^{(0)} \ , \quad z_6^{(1)} = \frac{z_6^{(0)}}{y_6^{(0)}} \ , \quad w_6^{(1)} = \frac{w_6^{(0)}}{y_6^{(0)}}.$$

**Step 2:** We blow up along the surface

$$\{(x_6^{(1)}, y_6^{(1)}, z_6^{(1)}, w_6^{(1)}) | x_6^{(1)} + t z_6^{(1)}/s - t = y_6^{(1)} = 0\}:$$

$$x_6^{(2)} = \frac{x_6^{(1)} + t z_6^{(1)}/s - t}{y_6^{(1)}} \ , \quad y_6^{(2)} = y_6^{(1)} \ , \quad z_6^{(2)} = z_6^{(1)} \ , \quad w_6^{(2)} = w_6^{(1)}.$$

**Step 3:** We blow up along the surface

$$\{(x_6^{(2)}, y_6^{(2)}, z_6^{(2)}, w_6^{(2)}) | x_6^{(2)} - \alpha_6 = y_6^{(2)} = 0\}:$$

$$x_6^{(3)} = \frac{x_6^{(2)} - \alpha_6}{y_6^{(2)}} \ , \quad y_6^{(3)} = y_6^{(2)} \ , \quad z_6^{(3)} = z_6^{(2)} \ , \quad w_6^{(3)} = w_6^{(2)}.$$

We have resolved the accessible singular point $P_6$.

By choosing a new coordinate system as

$$(x_6, y_6, z_6, w_6) = (-x_6^{(3)}, y_6^{(3)}, z_6^{(3)}, w_6^{(3)}),$$

we can obtain the coordinate system $(x_6, y_6, z_6, w_6)$ given in Theorem 3.1.

For the cases of $P_4, P_5$, the proof is similar.

The proof has thus been completed. ∎

## 5. New expression of the system (11)

In the case of holomorphy conditions $(x_i, y_i, z_i, w_i)$ $(i = 1, 2, \dots, 6)$, each coordinate system is classified as follows:

| | |
|---|---|
| Type (C,C,S) | $(x_i, y_i, z_i, w_i)$ $(i = 1, 2)$ |
| Type (P,S,S) | $(x_i, y_i, z_i, w_i)$ $(i = 3, 4, 5, 6)$ |

We denote that P is a point, C is a curve and S is a surface (see Figure 2). These properties suggest the possibility that there suggests a procedure for searching for other versions with different types of accessible singularities from the system (11). Now, we consider the following problem.

**Problem 5.1.** Can we find a polynomial Hamiltonian system with Hamiltonians $H_i \in \mathbb{C}(t, s)[q_1, p_1, q_2, p_2]$ $(i = 1, 2)$ with different types of accessible singularities from the system (11)?

To answer this, we construct the polynomial Hamiltonian system (2) with different type of holomorphy conditions of the system (11).

**Theorem 5.1.** *Let us consider a polynomial Hamiltonian system with Hamiltonians $H_i \in \mathbb{C}(t, s)[q_1, p_1, q_2, p_2]$ $(i = 1, 2)$. We assume that*

*(A1) $deg(H_i) = 5$ with respect to $q_1, p_1, q_2, p_2$.*

*(A2) This system becomes again a polynomial Hamiltonian system in each coordinate $r'_j$, $j = 1, 2, \dots, 6$:*

(17)
$$
\begin{aligned}
r'_1 :& x_1 = \frac{1}{q_1},\ y_1 = -q_1(q_1p_1 + \alpha_2),\ z_1 = q_2,\ w_1 = p_2,\\
r'_2 :& x_2 = \frac{1}{q_1},\ y_2 = -q_1(q_1p_1 + q_2p_2 - \alpha_1),\ z_2 = \frac{q_2}{q_1},\ w_2 = q_1p_2,\\
r'_3 :& x_3 = q_1,\ y_3 = p_1,\ z_3 = \frac{1}{q_2},\ w_3 = -(q_2p_2 + \alpha_4)q_2,\\
r'_4 :& x_4 = -(q_1p_1 + q_2p_2 - (\alpha_1 + \alpha_3))p_1,\ y_4 = \frac{1}{p_1},\ z_4 = q_2p_1,\ w_4 = \frac{p_2}{p_1},\\
r'_5 :& x_5 = -((q_1 - 1)p_1 + (q_2 - 1)p_2 - (\alpha_1 + \alpha_5))p_1,\ y_5 = \frac{1}{p_1},\\
& z_5 = (q_2 - 1)p_1,\ w_5 = \frac{p_2}{p_1},\\
r'_6 :& x_6 = -((q_1 - t)p_1 + (q_2 - s)p_2 - (\alpha_1 + \alpha_6))p_1,\ y_6 = \frac{1}{p_1},\\
& z_6 = (q_2 - s)p_1,\ w_6 = \frac{p_2}{p_1}.
\end{aligned}
$$

*Then such a system coincides with the system* (2).

We remark that each transformation of each coordinate $r'_i$ $(i = 1, 2, \dots, 6)$ is birational and symplectic.

**Proposition 5.2.** *On each coordinate $r'_j$, $j = 1, 2, \dots, 6$, the Hamiltonians $H'_{j1}$ and $H'_{j2}$ on $U_j \times B$ are expressed as a polynomial in $x_j, y_j, z_j, w_j$ and a rational*

*function in $t$ and $s$, and satisfy the following conditions*:

(18)
$$\begin{aligned}
&dq_1 \wedge dp_1 + dq_2 \wedge dp_2 - dH_1' \wedge dt - dH_2' \wedge ds\\
&= dx_j \wedge dy_j + dz_j \wedge dw_j - dH_{j1}' \wedge dt - dH_{j2}' \wedge ds \quad (j = 1, 2, \dots, 5),\\
&dq_1 \wedge dp_1 + dq_2 \wedge dp_2 - d(H_1' - p_1) \wedge dt - d(H_2' - p_2) \wedge ds\\
&= dx_6 \wedge dy_6 + dz_6 \wedge dw_6 - dH_{61}' \wedge dt - dH_{62}' \wedge ds.
\end{aligned}$$

**Proof of Theorem 5.1.** For the transformations $r_i'$ $(i = 1, 2, \dots, 5)$ we can repeat the same way given in Theorem 3.1.

We must be careful of the case of the transformation $r_6'$. The relation between the coordinate system $(x_6, y_6, z_6, w_6)$ and the coordinate system $(q_1, p_1, q_2, p_2)$ is given by

$$\begin{aligned}
&dx_6 \wedge dy_6 + dz_6 \wedge dw_6\\
&= dq_1 \wedge dp_1 + dq_2 \wedge dp_2 + dp_1 \wedge dt + dp_2 \wedge ds.
\end{aligned} \tag{19}$$

Resolving the coordinate $r_6$ in the variables $q_1, p_1, q_2, p_2$, we obtain

$(R_6)$
$$q_1 = t - x_6 y_6^2 - y_6 z_6 w_6 + (\alpha_1 + \alpha_6) y_6, \quad p_1 = 1/y_6, \quad q_2 = y_6 z_6 + s, \quad p_2 = w_6/y_6.$$

In this case, we must consider the polynomiality for $R_6(H_1 - p_1)$ and $R_6(H_2 - p_2)$, respectively.

In this way, we can obtain the Hamiltonians $H_1', H_2'$. ∎

In the case of holomorphy conditions $(x_i, y_i, z_i, w_i)$ $(i = 1, 2, \dots, 6)$, each coordinate system is classified as follows:

| | |
|---|---|
| Type (C,C,S) | $(x_i, y_i, z_i, w_i)$ $(i = 2, 4, 5, 6)$ |
| Type (P,S,S) | $(x_i, y_i, z_i, w_i)$ $(i = 1, 3)$ |

Moreover, we show that the system (11) is equivalent to the system (2) by giving an explicit rational and symplectic transformation. This transformation can be considered as a generalization of *Okamoto-transformation* of the sixth Painlevé system.

**Theorem 5.3.** *By using the rational and symplectic transformation* (1)*, the system* (11) *coincides with* (2)*.*

**Proof of Theorem 5.3** Set
$$X := q_1 + \frac{q_2 p_2 + \alpha_1}{p_1}, \quad Y := p_1, \quad Z := \frac{p_2}{p_1}, \quad W := -q_2 p_1 \quad T := t, \quad S := \frac{t}{s}.$$

By resolving in $q_1, p_1, q_2, p_2, t, s$, we obtain

$\tilde{S}$ :
$$q_1 = X + \frac{ZW - \alpha_1}{Y}, \quad p_1 = Y, \quad q_2 = -\frac{W}{Y}, \quad p_2 = YZ, \quad t = T, \quad s = \frac{T}{S}.$$

Applying the transformations in $t, s$ and the transformation of the symplectic 2-form:

$$\begin{aligned}
&dX \wedge dY + dZ \wedge dW = dq_1 \wedge dp_1 + dq_2 \wedge dp_2,\\
&dt = dT,\\
&ds = \frac{1}{S} dT - \frac{T}{S^2} dS,
\end{aligned} \tag{20}$$

we obtain the polynomial Hamiltonians $\tilde{S}(H_1+\frac{1}{S}H_2), \tilde{S}(-\frac{T}{S^2}H_2)$, which satisfy the following relations:

$$\tilde{S}(H_1+\frac{1}{S}H_2)=H_1', \quad \tilde{S}(-\frac{T}{S^2}H_2)=H_2'.$$

This completes the proof. ▮

We also consider the inverse transformation of (1).

**Lemma 5.4.** *The transformation*

$$R:(q_1,p_1,q_2,p_2,t,s)\to\left(q_1+\frac{q_2p_2-\alpha_1}{p_1},p_1,-\frac{p_2}{p_1},q_2p_1,t,\frac{t}{s}\right) \tag{21}$$

*satisfy the following relations*:

$$R\circ S=1, \quad S\circ R=1. \tag{22}$$

Composing such transformations, we can make a Bäcklund transformation.

**Proposition 5.5.** *By using the transformations $S$ and*

$$R':(q_1,p_1,q_2,p_2,t,s)\to\left(-\frac{p_1}{p_2},q_1p_2,q_2+\frac{q_1p_1-\alpha_1}{p_2},p_2,\frac{s}{t},s\right), \tag{23}$$

*we make the Bäcklund transformation $R'\circ S$ of* (11):

$$\begin{aligned} R'\circ S :&(q_1,p_1,q_2,p_2,t,s;\alpha_1,\dots,\alpha_6)\to\\ &\left(\frac{1}{q_2},-(q_1p_1+q_2p_2+\alpha_1)q_2,-\frac{q_1}{q_2},-p_1q_2,\frac{1}{s},\frac{t}{s};\alpha_1,\alpha_4,\alpha_2,\alpha_3,\alpha_5,\alpha_6\right).\end{aligned}$$

**Proposition 5.6.** *By using the transformations $R$ and*

$$\begin{aligned} N:(q_1,p_1,q_2,p_2t,s)\to(&\frac{p_1}{q_1p_1+q_2p_2+\alpha_1},-q_1(q_1p_1+q_2p_2+\alpha_1),\\ &\frac{p_2}{q_1p_1+q_2p_2+\alpha_1},-q_2(q_1p_1+q_2p_2+\alpha_1),\frac{1}{t},\frac{1}{s}),\end{aligned}$$

*we make the Bäcklund transformation $R\circ N$ of* (2):

$$\begin{aligned} R\circ N :&(q_1,p_1,q_2,p_2,t,s;\alpha_1.\dots,\alpha_6)\to\\ &\left(\frac{1}{q_1},-(q_1p_1+q_2p_2+\alpha_1)q_1,-\frac{q_2}{q_1},-p_2q_1,\frac{1}{t},\frac{s}{t};\alpha_1,\alpha_3,\alpha_2,\alpha_4,\alpha_5,\alpha_6\right).\end{aligned}$$

Now, we consider the following problem.

**Problem 5.2.** For the system (2), can we find such a $S$-transformation different from the transformation $R$?

To answer this, we find the following rational and symplectic transformation.

**Theorem 5.7.** *By using the rational and symplectic transformation $S_1$*

(24)

$$S_1:(q_1,p_1,q_2,p_2,t,s)\to\left(q_1,p_1+\frac{q_2p_2-\alpha_1-\alpha_3}{q_1},\frac{q_2}{q_1},p_2q_1,t,\frac{s}{t}\right),$$

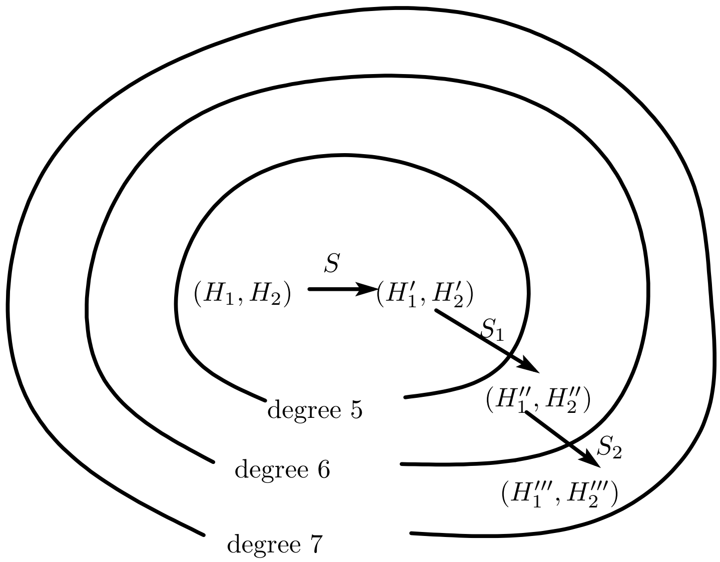


FIGURE 4. We note that the degree of each $H_i$ $(i=1,2)$ is 5 with respect to $q_1, p_1, q_2, p_2$.

*the system* (2) *is transformed into the Hamiltonian system*

(25)

$$\begin{aligned}
&dq_1 = \frac{\partial H_1''}{\partial p_1}dt + \frac{\partial H_2''}{\partial p_1}ds, \quad dp_1 = -\frac{\partial H_1''}{\partial q_1}dt - \frac{\partial H_2''}{\partial q_1}ds,\\
&dq_2 = \frac{\partial H_1''}{\partial p_2}dt + \frac{\partial H_2''}{\partial p_2}ds, \quad dp_2 = -\frac{\partial H_1''}{\partial q_2}dt - \frac{\partial H_2''}{\partial q_2}ds,\\
&H_1'' = H_{VI}(q_1, p_1, t; \alpha_1+\alpha_4+\alpha_6, \alpha_1+\alpha_2, \alpha_3, \alpha_1+\alpha_5, -\alpha_1-\alpha_3)\\
&\quad - \frac{\alpha_4}{t(ts-1)}q_1p_1 + \frac{\alpha_3(s-1)}{(t-1)(ts-1)}q_2p_2 + \frac{s}{(ts-1)}p_1p_2 - \frac{1}{t-1}p_1q_2p_2\\
&\quad + \frac{2(s-1)}{(t-1)(ts-1)}q_1p_1q_2p_2 - \frac{q_1q_2(q_1p_1+\alpha_3)\{(ts-1)p_2-(t-1)(q_2p_2+\alpha_4)\}}{t(t-1)(ts-1)},\\
&H_2'' = \pi(H_1''),
\end{aligned}$$

*where the transformation* $\pi$ *is explicitly given by*

(26)

$$\begin{aligned}
&\pi : (q_1, p_1, q_2, p_2, t, s; \alpha_1, \alpha_2, \dots, \alpha_6)\\
&\to (q_2, p_2, q_1, p_1, s, t; -\alpha_1-\alpha_2-\alpha_3-\alpha_4, \alpha_2, \alpha_4, \alpha_3, 1-\alpha_6, 1-\alpha_5).
\end{aligned}$$

We note that each degree of each of Hamiltonians $H_1'', H_2''$ is 6 with respect to $q_1, p_1, q_2, p_2$.

We also remark that on the polynomiality of the transformed Hamiltonian by $S_1$, the change of time-variables $t, s$ of the transformation $S_1$ is not essential.

**Proof of Theorem 5.7** Set

$$X := q_1, \quad Y := p_1 + \frac{q_2p_2-\alpha_1-\alpha_3}{q_1}, \quad Z := \frac{q_2}{q_1}, \quad W := q_1p_2 \quad T := t, \quad S := \frac{s}{t}.$$

By resolving in $q_1, p_1, q_2, p_2, t, s$, we obtain

$\tilde{S}_1$ :

$$q_1 = X, \quad p_1 = Y - \frac{ZW - \alpha_1 - \alpha_3}{X}, \quad q_2 = XZ, \quad p_2 = \frac{W}{X}, \quad t = T, \quad s = TS.$$

Applying the transformations in $t, s$ and the transformation of the symplectic 2-form:

$$\begin{aligned} &dX \wedge dY + dZ \wedge dW = dq_1 \wedge dp_1 + dq_2 \wedge dp_2, \\ &dt = dT, \\ &ds = SdT + TdS, \end{aligned} \tag{27}$$

we obtain the polynomial Hamiltonians $\tilde{S}_1(H_1' + SH_2'), \tilde{S}_1(TH_2')$, which satisfy the following relations:

$$\tilde{S}_1(H_1' + SH_2') = H_1'', \quad \tilde{S}_1(TH_2') = H_2''.$$

This completes the proof. ▮

**Theorem 5.8.** *Let us consider a polynomial Hamiltonian system with Hamiltonians $H_i \in \mathbb{C}(t,s)[q_1, p_1, q_2, p_2]$ $(i = 1, 2)$. We assume that*

*$(A1)$ $deg(H_i) = 6$ with respect to $q_1, p_1, q_2, p_2$.*

*$(A2)$ This system becomes again a polynomial Hamiltonian system in each coordinate $r_j''$, $j = 1, 2, \ldots, 6$:*

(28)

$$\begin{aligned}
r_1'' :& x_1 = -(q_1p_1 - q_2p_2 + \alpha_1 + \alpha_3)p_1, \ y_1 = \frac{1}{p_1}, \ z_1 = \frac{q_2}{p_1}, \ w_1 = p_2p_1, \\
r_2'' :& x_2 = \frac{1}{q_1}, \ y_2 = -q_1(q_1p_1 - q_2p_2 + \alpha_1 + \alpha_2 + \alpha_3), \ z_2 = q_2q_1, \ w_2 = \frac{p_2}{q_1}, \\
r_3'' :& x_3 = \frac{1}{q_1}, \ y_3 = -(p_1q_1 + \alpha_3)q_1, \ z_3 = q_2, \ w_3 = p_2, \\
r_4'' :& x_4 = q_1, \ y_4 = p_1, \ z_4 = \frac{1}{q_2}, \ w_4 = -(p_2q_2 + \alpha_4)q_2, \\
r_5'' :& x_5 = -((q_1 - 1)p_1 + (q_2 - 1)p_2 - (\alpha_1 + \alpha_5))p_1, \ y_5 = \frac{1}{p_1}, \\
& z_5 = (q_2 - 1)p_1, \ w_5 = \frac{p_2}{p_1}, \\
r_6'' :& x_6 = -((q_1 - t)p_1 + (q_2 - s)p_2 - (\alpha_1 + \alpha_6))p_1, \ y_6 = \frac{1}{p_1}, \\
& z_6 = (q_2 - s)p_1, \ w_6 = \frac{p_2}{p_1}.
\end{aligned}$$

*Then such a system coincides with the system* (25).

We remark that each transformation of each coordinate $r_i''$ $(i = 1, 2, \ldots, 6)$, is birational and symplectic.

The difference between the conditions $r_i'$ and the conditions $r_i''$ is type of accessible singularities. More precisely, we will explain in the next section.

**Proposition 5.9.** *On each coordinate $r_j''$, $j = 1, 2, \ldots, 6$, each of the Hamiltonians $H_{j1}''$ and $H_{j2}''$ on $U_j \times B$ is expressed as a polynomial in $x_j, y_j, z_j, w_j$ and a rational*

*function in $t$ and $s$, and satisfy the following conditions*:

(29)
$$\begin{aligned}
&dq_1 \wedge dp_1 + dq_2 \wedge dp_2 - dH_1'' \wedge dt - dH_2'' \wedge ds \\
&= dx_j \wedge dy_j + dz_j \wedge dw_j - dH_{j1}'' \wedge dt - dH_{j2}'' \wedge ds \quad (j = 1, 2, \dots, 5), \\
&dq_1 \wedge dp_1 + dq_2 \wedge dp_2 - d(H_1'' - p_1) \wedge dt - d(H_2'' - p_2) \wedge ds \\
&= dx_6 \wedge dy_6 + dz_6 \wedge dw_6 - dH_{61}'' \wedge dt - dH_{62}'' \wedge ds.
\end{aligned}$$

We also consider the inverse transformation of $S_1$.

**Lemma 5.10.** *The transformation*

(30)
$$R_1 : (q_1, p_1, q_2, p_2, t, s) \to \left(q_1, p_1 - \frac{q_2 p_2 - \alpha_1 - \alpha_3}{q_1}, q_2 q_1, \frac{p_2}{q_1}, t, ts\right)$$

*satisfy the following relations*:

$$R_1 \circ S_1 = 1, \quad S_1 \circ R_1 = 1. \tag{31}$$

Now, we also consider the following problem.

**Problem 5.3.** For the system (25), can we find such a $S$-transformation different from the transformation $R_1$?

To answer this, we find the following rational and symplectic transformation.

**Theorem 5.11.** *By using the rational and symplectic transformation $S_2$*

(32)
$$S_2 : (q_1, p_1, q_2, p_2, t, s) \to \left(q_1 - \frac{q_2 p_2 - \alpha_1 - \alpha_2 - \alpha_3}{p_1}, p_1, \frac{q_2}{p_1}, p_2 p_1\right),$$

*the system (25) is transformed into a Hamiltonian system*

$$\begin{aligned}
dq_1 &= \frac{\partial H_1'''}{\partial p_1} dt + \frac{\partial H_2'''}{\partial p_1} ds, \quad dp_1 = -\frac{\partial H_1'''}{\partial q_1} dt - \frac{\partial H_2'''}{\partial q_1} ds, \\
dq_2 &= \frac{\partial H_1'''}{\partial p_2} dt + \frac{\partial H_2'''}{\partial p_2} ds, \quad dp_2 = -\frac{\partial H_1'''}{\partial q_2} dt - \frac{\partial H_2'''}{\partial q_2} ds
\end{aligned} \tag{33}$$

*with polynomial Hamiltonians $H_1''', H_2''' = \pi(H_1''')$, where the transformation $\pi$ is explicitly given by*

$$\begin{aligned}
&\pi : (q_1, p_1, q_2, p_2, t, s; \alpha_1, \alpha_2, \dots, \alpha_6) \\
&\to (-\frac{(q_1 p_1 - \alpha_2) p_1}{p_2}, \frac{p_2}{p_1}, q_2 + \frac{q_1 p_1 - \alpha_1 - \alpha_2 - \alpha_3}{p_2}, p_2, \\
&s, t; -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, \alpha_2, \alpha_4, \alpha_3, 1 - \alpha_6, 1 - \alpha_5).
\end{aligned} \tag{34}$$

We note that each degree of each of Hamiltonians $H_1''', H_2'''$ is 7 with respect to $q_1, p_1, q_2, p_2$.

**Theorem 5.12.** *Let us consider a polynomial Hamiltonian system with Hamiltonians $H_i \in \mathbb{C}(t, s)[q_1, p_1, q_2, p_2]$ $(i = 1, 2)$. We assume that*

*(A1) $deg(H_i) = 7$ with respect to $q_1, p_1, q_2, p_2$.*

*(A2) This system becomes again a polynomial Hamiltonian system in each coordinate* $r_j'''$, $j = 1, 2, \ldots, 6$:

(35)
$$\begin{aligned}
r_1''' :& x_1 = \frac{1}{q_1},\ y_1 = -(q_1p_1 + q_2p_2 - \alpha_1 - \alpha_2)q_1,\ z_1 = \frac{q_2}{q_1},\ w_1 = p_2q_1,\\
r_2''' :& x_2 = -(q_1p_1 - \alpha_2)p_1,\ y_2 = \frac{1}{p_1},\ z_2 = q_2,\ w_2 = p_2,\\
r_3''' :& x_3 = \frac{1}{q_1},\ y_3 = -(q_1p_1 + q_2p_2 - \alpha_1 - \alpha_2 - \alpha_3)q_1,\ z_3 = \frac{q_2}{q_1},\ w_3 = p_2q_1,\\
r_4''' :& x_4 = q_1,\ y_4 = p_1,\ z_4 = \frac{1}{q_2},\ w_4 = -(p_2q_2 + \alpha_4)q_2,\\
r_5''' :& x_5 = -((q_1 - 1)p_1 + 2(q_2 - \frac{1}{2p_1})p_2 - (2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5))p_1,\ y_5 = \frac{1}{p_1},\\
& z_5 = (q_2p_1 - 1)p_1,\ w_5 = \frac{p_2}{p_1^2},\\
r_6''' :& x_6 = -((q_1 - t)p_1 + 2(q_2 - \frac{s}{2p_1})p_2 - (2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_6))p_1,\ y_6 = \frac{1}{p_1},\\
& z_6 = (q_2p_1 - s)p_1,\ w_6 = \frac{p_2}{p_1^2}.
\end{aligned}$$

*Then such a system coincides with the system* (33).

We remark that each transformation of each coordinate $r_i'''$ $(i = 1, 2, \ldots, 6)$, is birational and symplectic.

**Proposition 5.13.** *On each coordinate* $r_j'''$, $j = 1, 2, \ldots, 6$, *each of the Hamiltonians* $H_{j1}'''$ *and* $H_{j2}'''$ *on* $U_j \times B$ *is expressed as a polynomial in* $x_j, y_j, z_j, w_j$ *and a rational function in* $t$ *and* $s$, *and satisfy the following conditions*:

(36)
$$\begin{aligned}
&dq_1 \wedge dp_1 + dq_2 \wedge dp_2 - dH_1''' \wedge dt - dH_2''' \wedge ds\\
&= dx_j \wedge dy_j + dz_j \wedge dw_j - dH_{j1}''' \wedge dt - dH_{j2}''' \wedge ds \quad (j = 1, 2, \ldots, 5),
\end{aligned}$$

(37)
$$\begin{aligned}
&dq_1 \wedge dp_1 + dq_2 \wedge dp_2 - d(H_1''' - p_1) \wedge dt - d(H_2''' - p_2/p_1) \wedge ds\\
&= dx_6 \wedge dy_6 + dz_6 \wedge dw_6 - dH_{61}''' \wedge dt - dH_{62}''' \wedge ds.
\end{aligned}$$

We also consider the inverse transformation of $S_2$.

**Lemma 5.14.** *The transformation*

$$R_2 : (q_1, p_1, q_2, p_2) \to \left(q_1 + \frac{q_2p_2 - \alpha_1 - \alpha_2 - \alpha_3}{p_1}, p_1, q_2p_1, \frac{p_2}{p_1}\right) \tag{38}$$

*satisfy the following relations*:

$$R_2 \circ S_2 = 1, \quad S_2 \circ R_2 = 1. \tag{39}$$

**Problem 5.4.** It is still an open question whether we classify such a $S$-transformation for each system.

## 6. On some Hamiltonian structures of the system (2)

In this section, we will explain the relation between the holomorphy conditions $r'_j$ with the accessible singularities of the system (2). Let us take the compactification

$$(q_1, p_1, q_2, p_2, t, s) \in \mathbb{C}^4 \times B_2 \text{ to } ([z_0 : z_1 : z_2 : z_3 : z_4], t, s) \in \mathbb{P}^4 \times B_2$$

with the natural embedding

$$(q_1, p_1, q_2, p_2) = (z_1/z_0, z_2/z_0, z_3/z_0, z_4/z_0).$$

Here $B_2 = \mathbb{C}^2 - \{t(t-1)s(s-1) = 0\}$. Fixing the parameters $\alpha_i$, consider the product $\mathbb{P}^4 \times B_2$ and extend the regular vector field on $\mathbb{C}^4 \times B_2$ to a rational vector field $\tilde{v}$ on $\mathbb{P}^4 \times B_2$. It is easy to see that $\mathbb{P}^4$ is covered by five copies of $\mathbb{C}^4$ by gluing the transformations (16). By the following lemma, we will show that each coordinate system $(x_i, y_i, z_i, w_i)$ $(i = 1, 2, \dots, 6)$ can be obtained by successive blowing-up procedures of the accessible singularities in the boundary divisor $\mathcal{H}(\cong \mathbb{P}^3) \subset \mathbb{P}^4$.

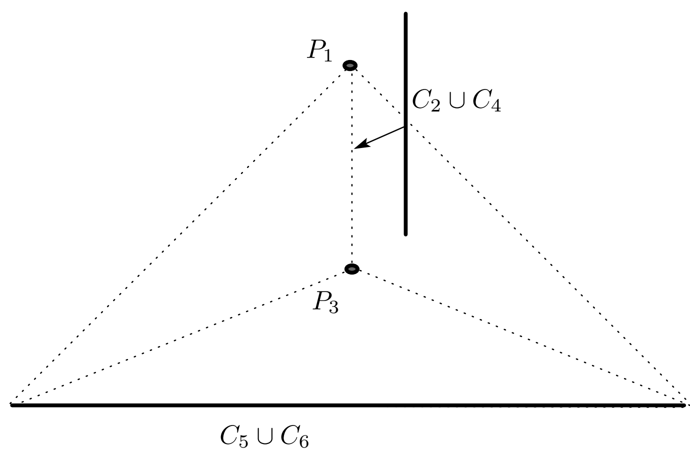


Figure 5. Accessible singularities of the system (2)

**Proposition 6.1.** *By resolving the following accessible singularities in the boundary divisor* $\mathcal{H}(\cong \mathbb{P}^3) \subset \mathbb{P}^4$

$$\begin{cases} P_1 = \{(X_1, Y_1, Z_1, W_1) | X_1 = Y_1 = Z_1 = W_1 = 0\}, \\ C_2 = \{(X_1, Y_1, Z_1, W_1) | X_1 = Y_1 = W_1 = 0\}, \\ P_3 = \{(X_2, Y_2, Z_2, W_2) | X_2 = Y_2 = Z_2 = W_2 = 0\}, \\ C_4 = \{(X_2, Y_2, Z_2, W_2) | Y_2 = Z_2 = W_2 = 0\}, \\ C_5 = \{(X_3, Y_3, Z_3, W_3) | X_3 = Y_3 = Z_3 = 0\}, \\ C_6 = \{(X_4, Y_4, Z_4, W_4) | X_4 = Z_4 = W_4 = 0\}, \end{cases} \tag{40}$$

*we can obtain the coordinate systems* $(x_i, y_i, z_i, w_i)$ $(i = 1, 2, \dots, 6)$*. Here* $C_2 \cup C_4 \cong \mathbb{P}^1$*,* $C_5 \cup C_6 \cong \mathbb{P}^1$*.*

**Proof of Proposition 6.1.** It is sufficient to show that we will resolve the accessible singularity $C_5 \cup C_6 \cong \mathbb{P}^1$. Other accessible singularities can be resolved as the same way in Proposition 4.1.

**Step 1:** We blow up along the curve $C_5 \cup C_6 \cong \mathbb{P}^1$:

$$x_1^{(1)} = \frac{X_3}{Y_3}\ , \quad y_1^{(1)} = Y_3\ , \quad z_1^{(1)} = \frac{Z_3}{Y_3}\ , \quad w_1^{(1)} = W_3,$$

$$x_2^{(1)} = \frac{X_4}{W_4}\ , \quad y_2^{(1)} = Y_4\ , \quad z_2^{(1)} = \frac{Z_4}{W_4}\ , \quad w_2^{(1)} = W_4.$$

It is easy to see that there are three accessible singularities

$$\begin{aligned}
L_1 =&\{(x_1^{(1)},y_1^{(1)},z_1^{(1)},w_1^{(1)})|x_1^{(1)}=y_1^{(1)}=z_1^{(1)}=0\} \cup \\
&\{(x_2^{(1)},y_2^{(1)},z_2^{(1)},w_2^{(1)})|x_2^{(1)}=z_2^{(1)}=w_2^{(1)}=0\} \cong \mathbb{P}^1,\\
L_2 =&\{(x_1^{(1)},y_1^{(1)},z_1^{(1)},w_1^{(1)})|x_1^{(1)}-1=y_1^{(1)}=z_1^{(1)}-1=0\} \cup \\
&\{(x_2^{(1)},y_2^{(1)},z_2^{(1)},w_2^{(1)})|x_2^{(1)}-1=z_2^{(1)}-1=w_2^{(1)}=0\} \cong \mathbb{P}^1,\\
L_3 =&\{(x_1^{(1)},y_1^{(1)},z_1^{(1)},w_1^{(1)})|x_1^{(1)}-t=y_1^{(1)}=z_1^{(1)}-s=0\} \cup \\
&\{(x_2^{(1)},y_2^{(1)},z_2^{(1)},w_2^{(1)})|x_2^{(1)}-t=z_2^{(1)}-s=w_2^{(1)}=0\} \cong \mathbb{P}^1.
\end{aligned}$$

**Step 2:** We blow up along the curve $L_1$:

$$\begin{aligned}
&x_1^{(2)} = \frac{x_1^{(1)}}{y_1^{(1)}}\ , \quad y_1^{(2)} = y_1^{(1)}\ , \quad z_1^{(2)} = \frac{z_1^{(1)}}{y_1^{(1)}}\ , \quad w_1^{(2)} = w_1^{(1)},\\
&x_2^{(2)} = \frac{x_2^{(1)}}{w_2^{(1)}}\ , \quad y_2^{(2)} = y_2^{(1)}\ , \quad z_2^{(2)} = \frac{z_2^{(1)}}{w_2^{(1)}}\ , \quad w_2^{(2)} = w_2^{(1)}.
\end{aligned}$$

**Step 3:** We blow up along the surface

$$\begin{aligned}
S_1 =&\{(x_1^{(2)},y_1^{(2)},z_1^{(2)},w_1^{(2)})|x_1^{(2)}+z_1^{(2)}w_1^{(2)}-(\alpha_1+\alpha_3)=y_1^{(2)}=0\} \cup \\
&\{(x_2^{(2)},y_2^{(2)},z_2^{(2)},w_2^{(2)})|z_2^{(2)}+x_2^{(2)}y_2^{(2)}-(\alpha_1+\alpha_3)=w_2^{(2)}=0\}:
\end{aligned}$$

$$\begin{aligned}
&x_1^{(3)} = \frac{x_1^{(2)}+z_1^{(2)}w_1^{(2)}-(\alpha_1+\alpha_3)}{y_1^{(2)}}\ , \quad y_1^{(3)} = y_1^{(2)}\ , \quad z_1^{(3)} = z_1^{(2)}\ , \quad w_1^{(3)} = w_1^{(2)},\\
&x_2^{(3)} = x_2^{(2)}\ , \quad y_2^{(3)} = y_2^{(2)}\ , \quad z_2^{(3)} = \frac{z_2^{(2)}+x_2^{(2)}y_2^{(2)}-(\alpha_1+\alpha_3)}{w_2^{(2)}}\ , \quad w_2^{(3)} = w_2^{(2)}.
\end{aligned}$$

We have resolved the accessible singularity $L_1$.

For the remaining accessible singularities, the proof is similar.

The proof has thus been completed. ∎

By the same way, we will explain the relation between the holomorphy conditions $r_j''$ with the following accessible singularities of the system (25).

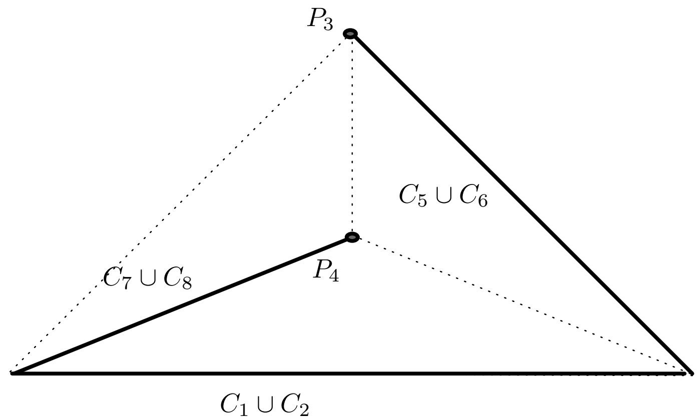


FIGURE 6. Accessible singularities of the system (25)

**Proposition 6.2.** *By resolving the following accessible singularities in the boundary divisor $\mathcal{H}(\cong \mathbb{P}^3) \subset \mathbb{P}^4$*

$$\begin{cases} C_1 = \{(X_3,Y_3,Z_3,W_3)|X_3=Y_3=Z_3=0\}, \\ C_2 = \{(X_4,Y_4,Z_4,W_4)|X_4=Z_4=W_4=0\}, \\ P_3 = \{(X_1,Y_1,Z_1,W_1)|X_1=Y_1=Z_1=W_1=0\}, \\ P_4 = \{(X_2,Y_2,Z_2,W_2)|X_2=Y_2=Z_2=W_2=0\}, \\ C_5 = \{(X_1,Y_1,Z_1,W_1)|X_1=Y_1=Z_1=0\}, \\ C_6 = \{(X_4,Y_4,Z_4,W_4)|Y_4=Z_4=W_4=0\}, \\ C_7 = \{(X_2,Y_2,Z_2,W_2)|X_2=Z_2=W_2=0\}, \\ C_8 = \{(X_3,Y_3,Z_3,W_3)|X_3=Y_3=W_3=0\}, \end{cases} \tag{41}$$

*we can obtain the coordinates $r_i''$ $(i=1,2,\dots,6)$. Here $C_1 \cup C_2 \cong \mathbb{P}^1$, $C_5 \cup C_6 \cong \mathbb{P}^1$ and $C_7 \cup C_8 \cong \mathbb{P}^1$.*

We remark that the relations between the accessible singularities given by Proposition 6.2 and the coordinates $r_j''$ $(j=1,2,\dots,6)$ are given as follows:

| $C_1 \cup C_2$ | $C_1 \cup C_2$ | $P_3$ | $P_4$ | $C_5 \cup C_6$ | $C_7 \cup C_8$ |
|---|---|---|---|---|---|
| $r_5''$ | $r_6''$ | $r_3''$ | $r_4''$ | $r_2''$ | $r_1''$ |

## 7. Invariant cycles of the system (11)

| codimension | invariant cycle | parameter's relation |
|---|---|---|
| 2 | $f_1^{(1)} := p_1, \quad f_1^{(2)} := p_2$ | $\alpha_1 = 0$ |
| 2 | $f_2^{(1)} := p_1, \quad f_2^{(2)} := p_2$ | $\alpha_1 = -\alpha_2$ |
| 1 | $f_3 := q_1$ | $\alpha_3 = 0$ |
| 1 | $f_4 := q_2$ | $\alpha_4 = 0$ |
| 1 | $f_5 := q_1 + q_2 - 1$ | $\alpha_5 = 0$ |
| 1 | $f_6 := q_1 + \frac{tq_2}{s} - t$ | $\alpha_6 = 0$ |

The list must be read as follows:

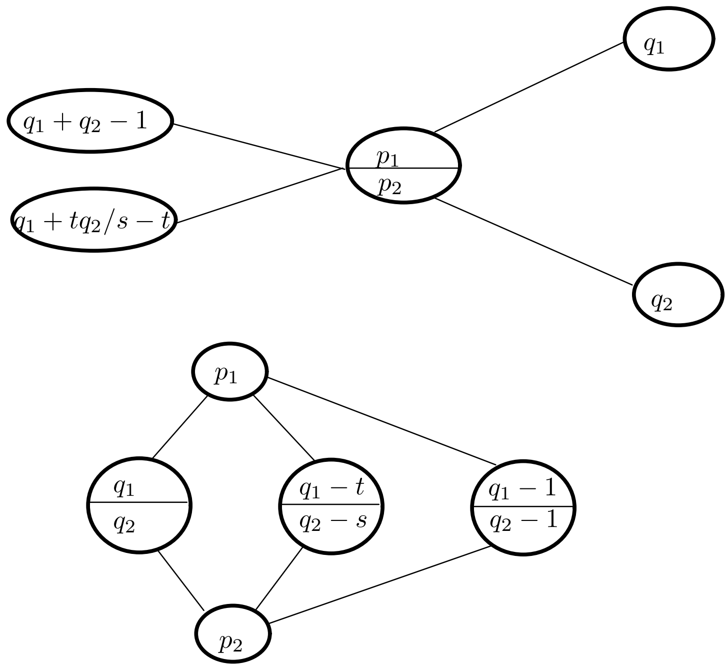


FIGURE 7. The symbol in each circle denotes the invariant cycle for each system.

Setting $\alpha_1 = 0$, then we see that the system (11) admits a particular solution expressed in terms of Appell's hypergeometric function:

(42)
$$
\begin{aligned}
&p_1 = p_2 = 0,\\
dq_1 =&\{-\frac{\alpha_3(q_1-1)(q_1-t)+(\alpha_4+\alpha_6-1)q_1(q_1-1)+\alpha_5 q_1(q_1-t)}{t(t-1)}\\
&-\frac{\alpha_4 s q_1}{t(t-s)}+\frac{\alpha_3 q_2}{t-s}\}dt+\{\frac{\alpha_2 q_1 q_2}{s(s-1)}+\frac{\alpha_3 t q_2}{s(s-t)}-\frac{\alpha_4(t-1)q_1}{(s-1)(s-t)}\}ds,\\
dq_2 =&\{\frac{\alpha_2 q_1 q_2}{t(t-1)}+\frac{\alpha_4 s q_1}{t(t-s)}-\frac{\alpha_3(s-1)q_2}{(t-1)(t-s)}\}dt\\
&+\{-\frac{\alpha_4(q_2-1)(q_2-s)+(\alpha_3+\alpha_6-1)q_2(q_2-1)+\alpha_5 q_2(q_2-s)}{s(s-1)}\\
&-\frac{\alpha_3 t q_2}{s(s-t)}+\frac{\alpha_4 q_1}{s-t}\}ds.
\end{aligned}
$$

This system is invariant under the transformation $\pi$:

$$\pi : (q_1, q_2, t, s; \alpha_2, \dots, \alpha_6) \to (q_2, q_1, s, t; \alpha_2, \alpha_4, \alpha_3, \alpha_5, \alpha_6).$$

This case corresponds to the slant part of Type A in Figure 6.
And, setting $\alpha_3 = 0$, then the system (11) admits a particular solution $q_1 = 0$. Moreover $(q_2, p_2)$ satisfy the sixth Painlevé system $H_{VI}$. And $p_1$ satisfies Riccati type equations whose coefficients are polynomials $(p_1, p_2)$, and so on.

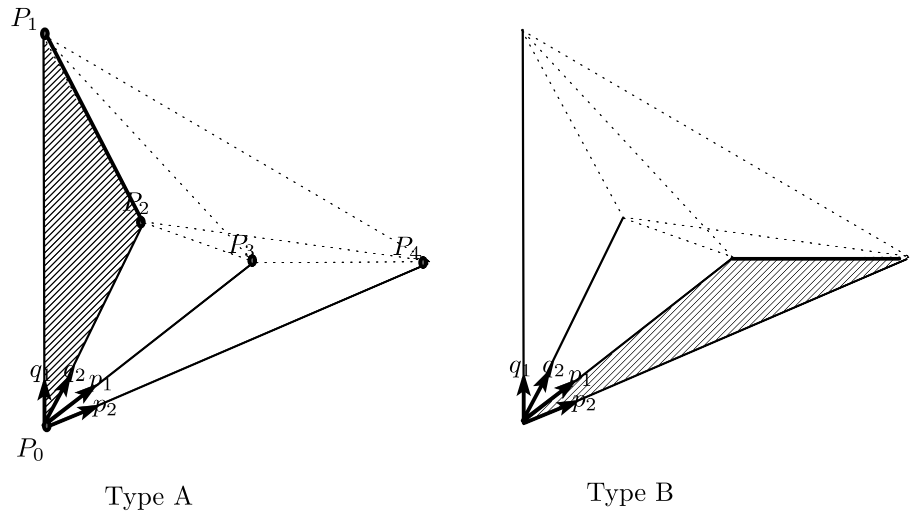


FIGURE 8. This figure denotes the four-dimensional projective space $\mathbb{P}^4 = \mathbb{C}^4 \sqcup \mathbb{P}^3$. $\mathbb{P}^4$ is covered by five open sets $\mathbb{C}^4$ around the points $P_i$ $(i = 0, 1, \dots, 4)$. We also remark that the figure spanned by the points $P_i$ $(i = 1, 2, \dots, 4)$ denotes the three-dimensional projective space $\mathbb{P}^3 = \mathbb{C}^3 \sqcup \mathbb{P}^2$, and spanned by the points $P_i$ $(i = 0, 1, 2)$ (resp. $P_i$ $(i = 0, 3, 4)$) denotes the two-dimensional projective space $\mathbb{P}^2 = \mathbb{C}^2 \sqcup \mathbb{P}^1$.

## 8. Invariant cycles of the system (2)

| codimension | invariant cycle | parameter's relation |
|---|---|---|
| 1 | $f_1 := p_1$ | $\alpha_2 = 0$ |
| 1 | $f_3 := p_2$ | $\alpha_4 = 0$ |
| 2 | $f_4^{(1)} := q_1, \ f_4^{(2)} := q_2$ | $\alpha_3 = -\alpha_1$ |
| 2 | $f_5^{(1)} := q_1 - 1, \ f_5^{(2)} := q_2 - 1$ | $\alpha_5 = -\alpha_1$ |
| 2 | $f_6^{(1)} := q_1 - t, \ f_6^{(2)} := q_2 - s$ | $\alpha_6 = -\alpha_1$ |
| 2 | $f_7^{(1)} := p_1, \ f_7^{(2)} := p_2$ | $\alpha_1 = 0$ |

Setting $\alpha_3 = -\alpha_1$, then we see that the system (2) admits a particular solution expressed in terms of Appell's hypergeometric function. This case corresponds to the slant part of Type B in Figure 6.

## 9. Bäcklund transformations of the system (11)

In this section, we study the symmetry of the Garnier system in two variables. The transformations $\pi_i$ $(i = 2, 3, \dots, 6)$ have been already obtained by H. Kimura (see [2]), and the transformations $w_i, \pi_1$ have been already obtained by T. Tsuda (see [22, 23, 24, 25].)

**Theorem 9.1.** (see [2, 22, 23, 24, 25]) The system (11) admits the following transformations as its Bäcklund transformations: with the notation $(*) = (q_1, p_1, q_2, p_2, t,$

$s;\alpha_1,\alpha_2,\dots,\alpha_6)$,

(43)

$$
\begin{aligned}
w_1:(*)\rightarrow&(q_1,p_1,q_2,p_2,t,s;\alpha_1+\alpha_2,-\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6),\\
w_2:(*)\rightarrow&(q_1,p_1-\frac{\alpha_3}{q_1},q_2,p_2,t,s;\alpha_1+\alpha_3,\alpha_2,-\alpha_3,\alpha_4,\alpha_5,\alpha_6),\\
w_3:(*)\rightarrow&(q_1,p_1,q_2,p_2-\frac{\alpha_4}{q_2},t,s;\alpha_1+\alpha_4,\alpha_2,\alpha_3,-\alpha_4,\alpha_5,\alpha_6),\\
w_4:(*)\rightarrow&(q_1,p_1-\frac{\alpha_5}{q_1+q_2-1},q_2,p_2-\frac{\alpha_5}{q_1+q_2-1},t,s;\\
&\alpha_1+\alpha_5,\alpha_2,\alpha_3,\alpha_4,-\alpha_5,\alpha_6),\\
w_5:(*)\rightarrow&(q_1,p_1-\frac{\alpha_6}{q_1+tq_2/s-t},q_2,p_2-\frac{\alpha_6 t}{s(q_1+tq_2/s-t)},t,s;\\
&\alpha_1+\alpha_6,\alpha_2,\alpha_3,\alpha_4,\alpha_5,-\alpha_6),\\
\pi_1:(*)\rightarrow&(\frac{p_1(q_1p_1-\alpha_3)}{(q_1p_1+q_2p_2+\alpha_1)(q_1p_1+q_2p_2+\alpha_1+\alpha_2)},\\
&-\frac{(q_1p_1+q_2p_2+\alpha_1)(q_1p_1+q_2p_2+\alpha_1+\alpha_2)}{p_1},\\
&\frac{p_2(q_2p_2-\alpha_4)}{(q_1p_1+q_2p_2+\alpha_1)(q_1p_1+q_2p_2+\alpha_1+\alpha_2)},\\
&-\frac{(q_1p_1+q_2p_2+\alpha_1)(q_1p_1+q_2p_2+\alpha_1+\alpha_2)}{p_2},\\
&1/t,1/s;-\alpha_1-\alpha_2-\alpha_3-\alpha_4,\alpha_2,\alpha_3,\alpha_4,1-\alpha_6,1-\alpha_5),\\
\pi_2:(*)\rightarrow&(1-q_1-q_2,-p_1,q_2,p_2-p_1,1-t,\frac{(t-1)s}{t-s};\alpha_1,\alpha_2,\alpha_5,\alpha_4,\alpha_3,\alpha_6),\\
\pi_3:(*)\rightarrow&(\frac{-sq_1-tq_2+ts}{(t-1)s},-(t-1)p_1,\frac{(t-s)q_2}{(t-1)s},-\frac{(t-1)(tp_1-sp_2)}{t-s},\\
&\frac{t}{t-1},\frac{t-s}{t-1};\alpha_1,\alpha_2,\alpha_6,\alpha_4,\alpha_5,\alpha_3),\\
\pi_4:(*)\rightarrow&(\frac{q_1}{t},tp_1,\frac{q_2}{s},sp_2,\frac{1}{t},\frac{1}{s},;\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_6,\alpha_5),\\
\pi_5:(*)\rightarrow&(-\frac{q_1}{q_2},-p_1q_2,\frac{1}{q_2}-(q_2p_2+q_1p_1+\alpha_1)q_2,\frac{t}{s},\frac{1}{s};\alpha_1,\alpha_4,\alpha_3,\alpha_2,\alpha_5,\alpha_6),\\
\pi_6:(*)\rightarrow&(q_2,p_2,q_1,p_1,s,t;\alpha_1,\alpha_2,\alpha_4,\alpha_3,\alpha_5,\alpha_6).
\end{aligned}
$$

The list (43) should be read as

$$
\begin{aligned}
&w_1(q_1)=q_1,\quad w_1(p_1)=p_1,\quad w_1(q_2)=q_2,\quad w_1(p_2)=p_2,\quad w_1(t)=t,\\
&w_1(\alpha_1)=\alpha_1+\alpha_2,\quad w_1(\alpha_2)=-\alpha_2,\quad w_1(\alpha_3)=\alpha_3,\\
&w_1(\alpha_4)=\alpha_4,\quad w_1(\alpha_5)=\alpha_5,\quad w_1(\alpha_6)=\alpha_6.
\end{aligned}
$$

**Lemma 9.2.** *The transformations described in Theorem 9.1 satisfy the following relations*:

$$
\begin{aligned}
&{w_1}^2={w_2}^2={w_3}^2={w_4}^2={w_5}^2=1, \quad {\pi_1}^2={\pi_2}^2={\pi_3}^2={\pi_4}^2={\pi_5}^2={\pi_6}^2=1,\\
&(w_1w_2)^2=(w_1w_3)^2=(w_1w_4)^2=(w_1w_5)^2=(w_2w_3)^2=(w_2w_4)^2\\
&=(w_2w_5)^2=(w_3w_4)^2=(w_3w_5)^2=(w_4w_5)^2=1,\\
&\pi_1(w_1,w_2,w_3)=(w_1,w_2,w_3)\pi_1, \quad (w_4\pi_1)^4=(w_5\pi_1)^4=1,\\
&\pi_2(w_1,w_2,w_3,w_4,w_5)=(w_1,w_4,w_3,w_2,w_5)\pi_2,\\
&\pi_3(w_1,w_2,w_3,w_4,w_5)=(w_1,w_5,w_3,w_4,w_2)\pi_3,\\
&\pi_4(w_1,w_2,w_3,w_4,w_5)=(w_1,w_2,w_3,w_5,w_4)\pi_4,\\
&\pi_5(w_1,w_2,w_3,w_4,w_5)=(w_3,w_2,w_1,w_4,w_5)\pi_5,\\
&\pi_6(w_1,w_2,w_3,w_4,w_5)=(w_1,w_3,w_2,w_4,w_5)\pi_6.
\end{aligned}
$$

In [18], T. Suzuki showed that the system (11) has affine Weyl group symmetry of type $B_5^{(1)}$, whose generators $s_i$ $(i=0,1,\dots,5)$ are explicitly given as follows: with the notation $(*)=(q_1,p_1,q_2,p_2,t,s;\gamma_0,\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5)$,

$$
(44)\quad
\begin{cases}
s_0(*) \rightarrow \Big(\dfrac{4p_1(q_1p_1-\gamma_4-\gamma_5)}{(2q_1p_1+2q_2p_2+\gamma_0-\gamma_3-2\gamma_4-3\gamma_5)(2q_1p_1+2q_2p_2+\gamma_0+\gamma_3-\gamma_5)},\\
\qquad -\dfrac{(2q_1p_1+2q_2p_2+\gamma_0-\gamma_3-2\gamma_4-3\gamma_5)(2q_1p_1+2q_2p_2+\gamma_0+\gamma_3-\gamma_5)}{4p_1},\\
\qquad \dfrac{4p_2(q_2p_2-\gamma_5)}{(2q_1p_1+2q_2p_2+\gamma_0-\gamma_3-2\gamma_4-3\gamma_5)(2q_1p_1+2q_2p_2+\gamma_0+\gamma_3-\gamma_5)},\\
\qquad -\dfrac{(2q_1p_1+2q_2p_2+\gamma_0-\gamma_3-2\gamma_4-3\gamma_5)(2q_1p_1+2q_2p_2+\gamma_0+\gamma_3-\gamma_5)}{4p_2},\\
\qquad \dfrac{1}{t},\dfrac{1}{s};-\gamma_0,\gamma_1,\gamma_2+\gamma_0,\gamma_3,\gamma_4,\gamma_5\Big),\\
s_1(*) \rightarrow \Big(\dfrac{q_1}{t},tp_1,\dfrac{q_2}{s},sp_2,\dfrac{1}{t},\dfrac{1}{s};\gamma_0,-\gamma_1,\gamma_2+\gamma_1,\gamma_3,\gamma_4,\gamma_5\Big),\\
s_2(*) \rightarrow \Big(\dfrac{q_1}{q_1+q_2-1},\left(p_1-(q_1p_1+q_2p_2+\dfrac{\gamma_0-\gamma_3-2\gamma_4-3\gamma_5}{2})\right)(q_1+q_2-1),\\
\qquad \dfrac{q_2}{q_1+q_2-1},\left(p_2-(q_1p_1+q_2p_2+\dfrac{\gamma_0-\gamma_3-2\gamma_4-3\gamma_5}{2})\right)(q_1+q_2-1),\\
\qquad \dfrac{t}{t-1},\dfrac{s}{s-1};\gamma_0+\gamma_2,\gamma_1+\gamma_2,-\gamma_2,\gamma_3+\gamma_2,\gamma_4,\gamma_5\Big),\\
s_3(*) \rightarrow \Big(\dfrac{1}{q_1},-(q_1p_1+q_2p_2+\dfrac{\gamma_0-\gamma_3-2\gamma_4-3\gamma_5}{2})q_1,-\dfrac{q_2}{q_1},-q_1p_2,\dfrac{1}{t},\dfrac{s}{t};\\
\qquad \gamma_0,\gamma_1,\gamma_2+\gamma_3,-\gamma_3,\gamma_4+\gamma_3,\gamma_5\Big),\\
s_4(*) \rightarrow (q_2,p_2,q_1,p_1,s,t;\gamma_0,\gamma_1,\gamma_2,\gamma_3+\gamma_4,-\gamma_4,\gamma_5+\gamma_4),\\
s_5(*) \rightarrow \Big(q_1,p_1,q_2,p_2-\dfrac{\gamma_5}{q_2},t,s;\gamma_0,\gamma_1,\gamma_2,\gamma_3,\gamma_4+2\gamma_5,-\gamma_5\Big).
\end{cases}
$$

Here, we give an explicit relation between root parameters $\gamma_0, \gamma_1, .., \gamma_5$ of type $B_5^{(1)}$ and the parameters $\alpha_1, \alpha_2, .., \alpha_6$ of the system (11) as follows:

(45)
$$(\gamma_0, \gamma_1, \dots, \gamma_5) = (1 - \alpha_5 - \alpha_6, -\alpha_5 + \alpha_6, -\alpha_2 + \alpha_5, \alpha_2 - \alpha_3, \alpha_3 - \alpha_4, \alpha_4).$$

**Lemma 9.3.** *The transformations described in* (44) *define a representation of the affine Weyl group of type $B_5^{(1)}$, that is, they satisfy the following relations*:

$$\begin{aligned}
&{s_0}^2 = {s_1}^2 = {s_2}^2 = {s_3}^2 = {s_4}^2 = {s_5}^2 = 1,\\
&(s_0s_1)^2 = (s_0s_3)^2 = (s_0s_4)^2 = (s_0s_5)^2 = (s_1s_3)^2 = (s_1s_4)^2 = (s_1s_5)^2 =\\
&(s_2s_4)^2 = (s_2s_5)^2 = (s_3s_5)^2 = 1,\\
&(s_0s_2)^3 = (s_1s_2)^3 = (s_2s_3)^3 = (s_3s_4)^3 = 1, \quad (s_4s_5)^4 = 1.
\end{aligned}$$

**Proposition 9.4.** *The affine Weyl group $< s_0, s_1, \dots, s_5 >$ of type $B_5^{(1)}$ given in* (44) *is equivalent to the group $< w_1, w_2, \dots, w_5, \pi_1, \pi_2, \dots, \pi_6 >$ given in* (43).

**Proof of Proposition 9.4.** We only have to check the correspondence between $s_0, s_1, \dots, s_5$ and $w_1, w_2, \dots, w_5, \pi_1, \pi_2, \dots, \pi_6$.

At first, we will show that

$$w_1, w_2, \dots, w_5, \pi_1, \pi_2, \dots, \pi_6 \in W(B_5^{(1)}).$$

The parameter's relations between $\alpha_i$ $(i = 1, 2, \dots, 6)$ and $\gamma_j$ $(j = 0, 1, \dots, 5)$ are given by

(46)
$$\begin{cases}
\alpha_1 = \dfrac{1 - \gamma_1 - 2\gamma_2 - 3\gamma_3 - 4\gamma_4 - 5\gamma_5}{2},\\
\alpha_2 = \gamma_3 + \gamma_4 + \gamma_5,\\
\alpha_3 = \gamma_4 + \gamma_5,\\
\alpha_4 = \gamma_5,\\
\alpha_5 = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5,\\
\alpha_6 = \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5.
\end{cases}$$

For each $w_1, w_2, \dots, w_5, \pi_1, \pi_2, \dots, \pi_6$, we obtain the following relations:

(47)
$$\begin{cases}
w_1 = s_3s_4s_5s_4s_3,\\
w_2 = s_4s_5s_4,\\
w_3 = s_5,\\
w_4 = s_3s_2s_3s_4s_5s_4s_3s_2s_3,\\
w_5 = s_1s_3s_2s_3s_4s_5s_4s_3s_2s_3s_1,\\
\pi_1 = s_0,\\
\pi_2 = s_4s_3s_4s_3s_4s_2s_3,\\
\pi_3 = s_4s_3s_4s_3s_4s_2s_3s_1s_4s_3s_4s_3s_4s_2s_3,\\
\pi_4 = s_1,\\
\pi_5 = s_4s_3s_4,\\
\pi_6 = s_4.
\end{cases}$$

Next, we will show that

$$s_0, s_1, \dots, s_5 \in < w_1, w_2, \dots, w_5, \pi_1, \pi_2, \dots, \pi_6 > .$$

The parameter's relations between $\gamma_j$ $(j = 0, 1, \dots, 5)$ and $\alpha_i$ $(i = 1, 2, \dots, 6)$ are given by

$$\begin{cases} \gamma_0 = 1 - \alpha_5 - \alpha_6, \\ \gamma_1 = -\alpha_5 + \alpha_6, \\ \gamma_2 = -\alpha_2 + \alpha_5, \\ \gamma_3 = \alpha_2 - \alpha_3, \\ \gamma_4 = \alpha_3 - \alpha_4, \\ \gamma_5 = \alpha_4. \end{cases} \tag{48}$$

For each $w_1, w_2, \dots, w_5, \pi_1, \pi_2, \dots, \pi_6$, we obtain the following relations:

$$\begin{cases} s_0 = \pi_1, \\ s_1 = \pi_4, \\ s_2 = \pi_5\pi_6\pi_5\pi_2\pi_6\pi_5\pi_6, \\ s_3 = \pi_6\pi_5\pi_6, \\ s_4 = \pi_6, \\ s_5 = w_3. \end{cases} \tag{49}$$

The proof has thus been completed. ▮

## 10. Bäcklund transformations of the system (2)

**Theorem 10.1.** *The system* (2) *admits the following transformations as its Bäcklund transformations*: with the notation $(*) = (q_1, p_1, q_2, p_2, t, s; \alpha_1, \alpha_2, \dots, \alpha_6)$,

$$\begin{aligned}
u_1 : (*) \rightarrow &(q_1 + \frac{\alpha_2}{p_1}, p_1, q_2, p_2, t, s; \alpha_1 + \alpha_2, -\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6), \\
u_2 : (*) \rightarrow &(q_1, p_1, q_2 + \frac{\alpha_4}{p_2}, p_2, t, s; \alpha_1 + \alpha_4, \alpha_2, \alpha_3, -\alpha_4, \alpha_5, \alpha_6), \\
u_3 : (*) \rightarrow &(\frac{q_1(q_1p_1 + q_2p_2 - \alpha_1)}{(q_1p_1 + q_2p_2 - \alpha_1 - \alpha_3)}, \frac{p_1(q_1p_1 + q_2p_2 - \alpha_1 - \alpha_3)}{(q_1p_1 + q_2p_2 - \alpha_1)}, \\
&\frac{q_2(q_1p_1 + q_2p_2 - \alpha_1)}{(q_1p_1 + q_2p_2 - \alpha_1 - \alpha_3)}, \frac{p_2(q_1p_1 + q_2p_2 - \alpha_1 - \alpha_3)}{(q_1p_1 + q_2p_2 - \alpha_1)}, \\
&t, s; \alpha_1 + \alpha_3, \alpha_2, -\alpha_3, \alpha_4, \alpha_5, \alpha_6), \\
u_4 : (*) \rightarrow &(\frac{(q_1 - 1)q_1p_1 + (q_2 - 1)q_1p_2 - \alpha_1 q_1 - \alpha_5}{(q_1 - 1)p_1 + (q_2 - 1)p_2 - \alpha_1 - \alpha_5}, \\
&\frac{\{(q_1 - 1)p_1 + (q_2 - 1)p_2 - \alpha_1 - \alpha_5\}p_1}{(q_1 - 1)p_1 + (q_2 - 1)p_2 - \alpha_1}, \\
&\frac{(q_1 - 1)p_1q_2 + (q_2 - 1)q_2p_2 - \alpha_1 q_2 - \alpha_5}{(q_1 - 1)p_1 + (q_2 - 1)p_2 - \alpha_1 - \alpha_5}, \\
&\frac{\{(q_1 - 1)p_1 + (q_2 - 1)p_2 - \alpha_1 - \alpha_5\}p_2}{(q_1 - 1)p_1 + (q_2 - 1)p_2 - \alpha_1}, \\
&t, s; \alpha_1 + \alpha_5, \alpha_2, \alpha_3, \alpha_4, -\alpha_5, \alpha_6),
\end{aligned} \tag{50}$$

$$
\begin{aligned}
u_5 : (*) \rightarrow (&\frac{(q_1-t)q_1p_1+(q_2-s)q_1p_2-\alpha_1q_1-\alpha_6t}{(q_1-t)p_1+(q_2-s)p_2-\alpha_1-\alpha_6},\\
&\frac{\{(q_1-t)p_1+(q_2-s)p_2-\alpha_1-\alpha_6\}p_1}{(q_1-t)p_1+(q_2-s)p_2-\alpha_1},\\
&\frac{(q_1-t)p_1q_2+(q_2-s)q_2p_2-\alpha_1q_2-\alpha_6s}{(q_1-t)p_1+(q_2-s)p_2-\alpha_1-\alpha_6},\\
&\frac{\{(q_1-t)p_1+(q_2-s)p_2-\alpha_1-\alpha_6\}p_2}{(q_1-t)p_1+(q_2-s)p_2-\alpha_1},\\
&t,s;\alpha_1+\alpha_6,\alpha_2,\alpha_3,\alpha_4,\alpha_5,-\alpha_6),\\
\varphi_1 : (*) \rightarrow (&\frac{1}{q_1},-(q_1p_1+\alpha_2)q_1,\frac{1}{q_2},-(q_2p_2+\alpha_4)q_2,\frac{1}{t},\frac{1}{s};\\
&-\alpha_1-\alpha_2-\alpha_3-\alpha_4,\alpha_2,\alpha_3,\alpha_4,1-\alpha_6,1-\alpha_5),\\
\varphi_2 : (*) \rightarrow (&1-q_1,-p_1,1-q_2,-p_2,1-t,1-s;\alpha_1,\alpha_2,\alpha_5,\alpha_4,\alpha_3,\alpha_6),\\
\varphi_3 : (*) \rightarrow (&\frac{t-q_1}{t-1},-(t-1)p_1,\frac{s-q_2}{s-1},-(s-1)p_2,\\
&\frac{t}{t-1},\frac{s}{s-1};\alpha_1,\alpha_2,\alpha_6,\alpha_4,\alpha_5,\alpha_3),\\
\varphi_4 : (*) \rightarrow (&\frac{q_1}{t},tp_1,\frac{q_2}{s},sp_2,\frac{1}{t},\frac{1}{s};\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_6,\alpha_5),\\
\varphi_5 : (*) \rightarrow (&q_2,p_2,q_1,p_1,s,t;\alpha_1,\alpha_4,\alpha_3,\alpha_2,\alpha_5,\alpha_6),\\
\varphi_6 : (*) \rightarrow (&\frac{q_1}{q_2},p_1q_2,\frac{1}{q_2},-(q_2p_2+q_1p_1-\alpha_1)q_2,\frac{t}{s},\frac{1}{s};\alpha_1,\alpha_2,\alpha_4,\alpha_3,\alpha_5,\alpha_6),\\
\varphi_7 : (*) \rightarrow (&\frac{t}{q_1},-\frac{(q_1p_1+\alpha_2)q_1}{t},\frac{s}{q_2},-\frac{(q_2p_2+\alpha_4)q_2}{s},t,s;\\
&-\alpha_1-\alpha_2-\alpha_3-\alpha_4,\alpha_2,\alpha_3,\alpha_4,1-\alpha_5,1-\alpha_6),\\
\varphi_8 : (*) \rightarrow (&\frac{(q_1-1)t}{q_1-t},-\frac{p_1(q_1-t)^2+\alpha_2(q_1-t)}{t(t-1)},\\
&\frac{(q_2-1)s}{q_2-s},-\frac{p_2(q_2-s)^2+\alpha_4(q_2-s)}{s(s-1)},\\
&t,s;\alpha_1+\alpha_3+\alpha_5-1,\alpha_2,1-\alpha_3,\alpha_4,1-\alpha_5,\alpha_6).
\end{aligned}
$$

**Remark 10.2.** The transformations $u_4, u_5, \varphi_7$ and $\varphi_8$ satisfy the following relations: .

$$
\begin{aligned}
&u_4 = \varphi_2 \circ s_3 \circ \varphi_2, \quad u_5 = \varphi_3 \circ s_3 \circ \varphi_3,\\
&\varphi_7 = \varphi_1 \circ \varphi_4, \quad \varphi_8 = \varphi_3 \circ \varphi_7 \circ \varphi_3.
\end{aligned} \tag{51}
$$

We note that by the transformations $S$ and $R$ (see Section 4) the relations between the transformations in Theorem 9.1 and the transformations in Theorem 10.1 are given as follows:

$$
\begin{aligned}
&S \circ w_i \circ R = u_i \quad (i = 1, 2, \dots, 5),\\
&S \circ \pi_i \circ R = \varphi_i \quad (i = 1, 2, \dots, 6).
\end{aligned} \tag{52}
$$

**Proposition 10.3.** *Let us define the following translation operators*:

$$
\begin{aligned}
T_1 :&= \varphi_4 s_4 \varphi_1 s_4,\\
T_2 :&= s_4 s_5 T_1 \varphi_1 s_4,\\
T_3 :&= \varphi_2 T_1 \varphi_2,\\
T_4 :&= \varphi_3 T_1 \varphi_3,\\
T_5 :&= \varphi_3 s_4 s_5 T_1 \varphi_1 s_4 \varphi_3,\\
T_6 :&= \varphi_4 s_4 s_5 T_1 \varphi_1 s_4 \varphi_4,\\
T_7 :&= \varphi_1 s_4 s_5 \varphi_4,\\
T_8 :&= \varphi_2 s_4 s_5 \varphi_4 \varphi_1 \varphi_2,\\
T_9 :&= \varphi_3 s_4 s_5 \varphi_4 \varphi_1 \varphi_3.
\end{aligned} \tag{53}
$$

These translation operators act on parameters as follows:

$$
\begin{aligned}
T_1(\alpha_1,\alpha_2,\dots,\alpha_6) =&(\alpha_1,\alpha_2,\dots,\alpha_6)+(0,0,0,0,-1,1),\\
T_2(\alpha_1,\alpha_2,\dots,\alpha_6) =&(\alpha_1,\alpha_2,\dots,\alpha_6)+(-1,0,0,0,0,2),\\
T_3(\alpha_1,\alpha_2,\dots,\alpha_6) =&(\alpha_1,\alpha_2,\dots,\alpha_6)+(0,0,-1,0,0,1),\\
T_4(\alpha_1,\alpha_2,\dots,\alpha_6) =&(\alpha_1,\alpha_2,\dots,\alpha_6)+(0,0,1,0,-1,0),\\
T_5(\alpha_1,\alpha_2,\dots,\alpha_6) =&(\alpha_1,\alpha_2,\dots,\alpha_6)+(-1,0,2,0,0,0),\\
T_6(\alpha_1,\alpha_2,\dots,\alpha_6) =&(\alpha_1,\alpha_2,\dots,\alpha_6)+(-1,0,0,0,2,0),\\
T_7(\alpha_1,\alpha_2,\dots,\alpha_6) =&(\alpha_1,\alpha_2,\dots,\alpha_6)+(1,0,0,0,-1,-1),\\
T_8(\alpha_1,\alpha_2,\dots,\alpha_6) =&(\alpha_1,\alpha_2,\dots,\alpha_6)+(-1,0,1,0,0,1),\\
T_9(\alpha_1,\alpha_2,\dots,\alpha_6) =&(\alpha_1,\alpha_2,\dots,\alpha_6)+(-1,0,1,0,1,0).
\end{aligned} \tag{54}
$$

## 11. Algebraic solutions of the system (2)

It is known that one can get an algebraic solution of Painlevé VI equations by considering the fixed points with respect to a Bäcklund transformation corresponding to a Dynkin automorphism.

For example, consider the Dynkin diagram automorphism

$$\pi_{13}(q,p,t;\alpha_0,\alpha_1,\dots,\alpha_4) \rightarrow (\frac{t}{q}, -\frac{(qp+\alpha_2)q}{t}, t; \alpha_0,\alpha_3,\alpha_2,\alpha_1,\alpha_4). \tag{55}$$

By this transformation, the fixed solution is derived from

$$\alpha_0=\alpha_3, \quad \alpha_1=\alpha_4, \quad q=\frac{t}{q}, \quad p=-\frac{(qp+\alpha_2)q}{t}. \tag{56}$$

Then we obtain

$$(q,p)=(\pm\sqrt{t}, \mp\frac{\alpha_2}{2\sqrt{t}}). \tag{57}$$

Masuda showed that by applying the Bäcklund transformation (see [4]) defined by

$$T := \pi_{13} s_3 s_2 s_1, \tag{58}$$

we can obtain Umemura's solution with special parameters.

In this section, we consider an extension of these algebraic solutions for the system (2).

At first, we consider the Dynkin diagram automorphism $\varphi_7$. By this transformation, the fixed solution is derived from

$$\begin{aligned}&\alpha_5 = 1-\alpha_5, \quad \alpha_6 = 1-\alpha_6,\\ &q_1 = \frac{t}{q_1}, \quad p_1 = -\frac{(q_1p_1+\alpha_2)q_1}{t}, \quad q_2 = \frac{s}{q_2}, \quad p_2 = -\frac{(q_2p_2+\alpha_4)q_2}{s}.\end{aligned} \tag{59}$$

Then we obtain

$$\begin{aligned}(\alpha_1,\alpha_2,\dots,\alpha_6) &= \left(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\frac{1}{2},\frac{1}{2}\right),\\ (q_1,p_1,q_2,p_2) &= \left(\pm\sqrt{t},\mp\frac{\alpha_2}{2\sqrt{t}},\pm\sqrt{s},\mp\frac{\alpha_4}{2\sqrt{s}}\right)\end{aligned} \tag{60}$$

as a seed solution.

Applying the Bäcklund transformation defined by

$$T := \varphi_2\varphi_1\varphi_2 u_4 u_2 u_1, \tag{61}$$

we can obtain an extended Umemura's solution with special parameters. Here, the transformation $T$ is explicitly given as follows:

$$\begin{aligned}&T(q_1,p_1,q_2,p_2,t,s;\alpha_1,\alpha_2,\dots,\alpha_6) \to\\ &(\frac{(q_1-1)q_1p_1^2+(q_2-1)q_1p_1p_2-(\alpha_1-\alpha_2)q_1p_1}{((q_1-1)p_1+(q_2-1)p_2-\alpha_1)((q_1-1)p_1+\alpha_2)}\\ &+\frac{-\alpha_2(p_1+p_2)+\alpha_2q_2p_2-\alpha_5p_1-\alpha_1\alpha_2}{((q_1-1)p_1+(q_2-1)p_2-\alpha_1)((q_1-1)p_1+\alpha_2)},\\ &-\frac{(q_1-1)\{(q_1-1)p_1+(q_2-1)p_2-\alpha_1\}\{(q_1-1)p_1+\alpha_2\}}{(q_1-1)p_1+(q_2-1)p_2-\alpha_1-\alpha_5},\\ &\frac{q_2(q_2-1)p_2^2-(\alpha_1-\alpha_4)q_2p_2-\alpha_4(p_1+p_2)+\alpha_4q_1p_1}{((q_1-1)p_1+(q_2-1)p_2-\alpha_1)((q_2-1)p_2+\alpha_4)}\\ &+\frac{-\alpha_1\alpha_4-\alpha_5p_2+(q_1-1)p_1q_2p_2}{((q_1-1)p_1+(q_2-1)p_2-\alpha_1)((q_2-1)p_2+\alpha_4)},\\ &-\frac{(q_2-1)((q_1-1)p_1+(q_2-1)p_2-\alpha_1)((q_2-1)p_2+\alpha_4)}{(q_1-1)p_1+(q_2-1)p_2-\alpha_1-\alpha_5},\\ &\frac{t}{t-1},\frac{s}{s-1};-\alpha_1,-\alpha_2,1-\alpha_6,-\alpha_4,-\alpha_5,1-\alpha_3).\end{aligned} \tag{62}$$

Next, we consider the Dynkin diagram automorphism $\varphi_8$. By this transformation, the fixed solution is derived from

$$\begin{aligned}&\alpha_3 = 1-\alpha_3, \quad \alpha_5 = 1-\alpha_5,\\ &q_1 = \frac{(q_1-1)t}{(q_1-t)}, \quad p_1 = -\frac{p_1(q_1-t)^2+\alpha_2(q_1-t)}{t(t-1)},\\ &q_2 = \frac{(q_2-1)s}{(q_2-s)}, \quad p_2 = -\frac{p_2(q_2-s)^2+\alpha_4(q_2-s)}{s(s-1)}.\end{aligned} \tag{63}$$

Then we obtain
(64)

$$
\begin{aligned}
&(\alpha_1, \alpha_2, \dots, \alpha_6) = \left(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \frac{1}{2}, \frac{1}{2}\right),\\
&(q_1, p_1, q_2, p_2) = \left(t \mp \sqrt{t(t-1)}, \pm \frac{\alpha_2\sqrt{t(t-1)}}{2t(t-1)}, s \mp \sqrt{s(s-1)}, \pm \frac{\alpha_4\sqrt{s(s-1)}}{2s(s-1)}\right)
\end{aligned}
$$

as a seed solution.

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## References


[1] K. Fuji and T. Suzuki, *The sixth Painlevé equation arising from $D_4^{(1)}$ hierarchy*, J. Phys. A: Math. Gen. **39** (2006), 12073–12082.

[2] H. Kimura, *Uniform foliation associated with the Hamiltonian system $\mathcal{H}_n$*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **20** (1993), no. 1, 1–60.

[3] H. Kimura and K. Okamoto, *On the polynomial Hamiltonian structure of the Garnier systems*, J. Math. Pures Appl. **63** (1984), 129–146.

[4] T. Masuda, *On a Class of Algebraic Solutions to the Painlevé VI Equations, Its Determinant Formula and Coalescence Cascade*, Funkcial. Ekvac. **46** (2003), 121–171.

[5] T. Matano, A. Matumiya and K. Takano, *On some Hamiltonian structures of Painlevé systems, II*, J. Math. Soc. Japan **51** (1999), 843–866.

[6] M. Noumi and Y. Yamada, *Affine Weyl Groups, Discrete Dynamical Systems and Painlevè Equations*, Comm Math Phys **199** (1998), 281–295.

[7] M. Noumi and Y. Yamada, *A new Lax pair for the sixth Painlevé equation associated with* $\mathfrak{so}(8)$, in Microlocal Analysis and Complex Fourier Analysis. ed. T.Kawai and K.Fujita, (World Scientific, 2002), 238–252.

[8] M. Noumi, *Painlevé Equations Through Symmetry*, American Mathematical Society, **223** (2004).

[9] M.-H. Saito, *Deformation of logarithmic symplectic manifold and equations of Painlevé type*, in preparation.

[10] Y. Sasano, *Coupled Painlevé V systems in dimension* 4 , Funkcial. Ekvac. **49** (2006), 133–161.

[11] Y. Sasano, *The phase space of coupled Painlevé III system in dimension four*, to appear in Kyoto Journal.

[12] Y. Sasano, *Coupled Painlevé II systems in dimension four and the systems of type $A_4{}^{(1)}$*, to appear in Tohoku Journal.

[13] Y. Sasano, *Higher order Painlevé equations of type $D_l{}^{(1)}$*, RIMS Kokyuroku **1473** (2006), 143–163.

[14] Y. Sasano and Y. Yamada, *Symmetry and holomorphy of Painlevé type systems*, preprint.

[15] T. Shioda and K. Takano, *On some Hamiltonian structures of Painlevé systems I*, Funkcial. Ekvac. **40** (1997), 271–291.

[16] M. Suzuki, N. Tahara and K. Takano, *Hierarchy of Bäcklund transformation groups of the Painlevé equations*, J. Math. Soc. Japan **56** (2004), 1221–1232.

[17] M. Suzuki, *Spaces of initial conditions of Garnier system and its degenerate systems in two variables*, J. Math. Soc. Japan **58** (2006), 1079–1117.

[18] T. Suzuki, *Affine Weyl group symmetry of the Garnier system*, Funkcial. Ekvac. **48** (2005), 203–230.

[19] N. Tahara, *An augmentation of the phase space of the system of type $A_4^{(1)}$*, Kyushu J. Math. **58** (2004), 393–425.

[20] K. Takano, *Confluence processes in defining manifolds for Painlevé systems*, Tohoku Math. J. (2) **53** (2001), 319–335.

[21] T. Tsuda, K. Okamoto and H. Sakai, *Folding transformations of the Painlevé equations*, Math. Ann. **331** (2005), 713–738.

[22] T. Tsuda, *Birational Symmetries, Hirota Bilinear Forms and Special Solutions of the Garnier Systems in 2-variables*, J. Math .Sci. Univ. Tokyo. **10** (2003), 355–371.

[23] T. Tsuda, *Rational Solutions of the Garnier System in Terms of Schur Polynomials*, IMRN. **43** (2003), 2341–2358.
[24] T. Tsuda, *Universal characters and integrable systems*, PhD thesis. The University of Tokyo, (2003).
[25] T. Tsuda, *Toda equation and special polynomials associated with the Garnier system*, Advances in Mathematics. **206** (2006), 657–683.